Basic Statistics for the Health Sciences

Third Edition

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Glossary of Symbols

Symbols taken from letters of the English alphabet

Symbol	How it is read	What it means
a		Sample y intercept of a straight line of best fit
b		Sample regression coefficient, slope, or gradient of line of best fit
CI		Confidence interval
CV		Coefficient of variation
$C(N,r)$	C of N comma r	Combinations of N things taken r at a time
$\bar{d}$	d-bar	Mean difference of sample observations
df		Degrees of freedom
E		Expected value in a cell of contingency table
f		Frequency
H_a or H_1	H sub a or H sub one	Alternate hypothesis
H_0	H sub zero	Null hypothesis
n		Sample size
N		Population size
O		Observed value in a cell of contingency table
p		Probability of a success in a single binomial trial
p	p	Estimate of the proportion of individuals with a certain characteristic
p	p value	Probability of a statistic occurring by chance
$P(A)$	probability of A	Probability that event A occurs
P_i, P_{90}	P sub i, P sub ninety	ith percentile, 90th percentile
$P(N,r)$	P of N comma r	Permutations of N things taken r at a time
q		Probability of a failure in a single binomial trial
r		Sample correlation coefficient
s		Sample standard deviation
s^2	s squared	Sample variance
s_d	s sub d	Standard deviation of differences of sample observations
s_p	s sub p	Pooled standard deviation
$s_{\bar{x}}$	s sub x-bar	Sample standard error of the mean; standard deviation of sampling distribution
SE(b)	standard error of b	Standard error of the regression coefficient
SE ($\bar{x}$)	standard error of x-bar	Standard error of the mean (same as $s_{\bar{x}}$ or $\sigma_{\bar{x}}$)
E($p_1 - p_2$)	standard error of p-one minus p-two	Standard error of the difference of two proportions

ol	How it is read	What it means
$x_2)$	standard error of x-one-bar minus x-two-bar	Standard error of the difference of the two means
	t value	t value from Student's t distribution
	t sub alpha	t value corresponding to a specified tail area α
	X-bar	Sample mean; mean of x values
	Y-bar	Mean of y values
	Z score	Standard normal deviate

ols taken from letters of the Greek alphabet

	alpha	Significance level
)r	alpha error	Type I error in hypothesis testing
)r	beta error	Type II error in hypothesis testing
	alpha	y intercept of population regression line
	beta	Slope of population regression line
	chi-square (pronounced "ki-square")	Test statistic for contingency table
	delta	Mean difference of population observations
	delta (capital delta)	Δx means change in x
	mu	Population mean
	mu sub zero	Baseline value of μ
	mu sub x-bar	Mean of sampling distribution
	sigma	Population standard deviation
	sigma squared	Population variance
	sigma sub x bar	Standard error of the mean
	the sum of (capital sigma)	Sum the values that follow
	rho	Population correlation coefficient

iematical symbols

	absolute value of x	Take the numerical value of x, ignoring the sign
	n factorial	$n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1$
	greater than	Number on left is larger than number on right
	greater than or equal to	Number on left is larger than or equal to number on right
	less than	Number on left is smaller than number on right
	less than or equal to	Number on left is smaller than or equal to number on right
	not equal to	The two values on either side of the symbol are not the same value

Contents

Preface

Statistics is a peculiar subject. Unaccountably, many students who handle their toughest studies with aplomb view statistics as a nearly insurmountable barrier. Perhaps this derives from the inherent difficulty of viewing the world in probabilistic terms, or from the underlying mathematics, or from the often abstruse mode of presentation. I hope that *Basic Statistics for the Health Sciences* is a step toward overcoming these problems.

The purpose of this book is to present some of the concepts, principles, and methods of statistics in as clear and understandable a manner as possible. The level is appropriate to students with a limited mathematical background but for whom a working knowledge of statistics is indispensable. In my own teaching I have found this approach to be particularly effective with students of medicine, nursing, public health, and the allied health sciences.

Because statistics allows us to use data to gain insight into a problem, we emphasize understanding rather than mastering a statistical technique. Therefore, the underlying objective of this book is to introduce concepts intuitively rather than via rigorous mathematics.

Certain features of this book's organization have proven to be especially effective. These include—in each chapter—an outline, learning objectives (which may easily be used as review questions), highlighting of important terms, a concluding statement, and a list of newly introduced vocabulary. My colleagues and I have found these to be simple but effective aids for any student striving for mastery of the material. Nearly all the examples and exercises are adapted from actual data in health research, so students are quite likely to appreciate the relevance of the material to their chosen field. My own extended research is reflected in the recurring theme of these pages—the effect our lifestyle choices have on our health.

This text goes somewhat beyond the coverage of most elementary statistics books by including a number of special topics for students of different disciplines and interests. Some key principles of epidemiology are introduced; the topics of age-adjustment and relative risk are covered. A chapter on probability, often reserved for more sophisticated treatments, is included. By understanding probability, the student gains a better insight into several of the subsequent topics. There are chapters devoted to correlation, regression, and analysis of

variance, as well as to distribution-free methods, a subject that appears to be gaining favor rapidly. Chapters on vital statistics and life tables are included to meet the special needs of medical and public health students. Two important subjects—how to perform a health survey and how to evaluate a research report—should be of real benefit to those who will carry out or use the results of research projects.

This text can be used for a course of three quarter units or three semester units, depending on the topics the instructor chooses to emphasize. It contains the material I have found to be uncommonly effective in motivating students' interest in statistics, so they begin to see it as a very satisfying form of detective work. The book should be especially useful for the student who enters the course with some lingering doubts about his or her ability to master statistics or for the student who initially questions the relevance of studying the subject.

Changes in the Third Edition

We have been delighted with the positive reception of this text by individuals from diverse institutions across the country—students, instructors, and reviewers. Clearly, the strong public health and health sciences emphasis of this text has become an outstanding feature of this book. This third edition builds on the philosophy and pedagogic approaches of the first two editions. It includes improvements that we hope will make this new edition even more useful in meeting the teaching objectives of instructors and will make it even easier for students to comprehend. Major changes for the third edition include the following:

- The number and range of exercises have been expanded. Each chapter now contains several additional exercises, some using large data sets.
- Several topics have been added, including Bayes theorem, Kruskal–Wallis ANOVA, and box and whisker plot.
- Most of the chapters and selected end-of-chapter exercises have been revised and updated.

My experience with the second edition of the text has been gratifying. May your experience with this new edition be gratifying as well!

Writing a textbook is a labor of love—characterized by both pleasure and agony. Its completion follows the convergence of a number of factors: the idea, the encouragement of friends and colleagues, and the cooperation of assistants. I wish especially to acknowledge the inspiring influence of my teachers—John W. Fertig, Chin Long Chiang, and Richard D. Remington—and my mentor, Wilfred J. Dixon. They demonstrated to me that statistics, an often abstruse subject, can indeed be taught in a clear and understandable fashion. The manuscript benefited from the professional insights of Paul S. Anderson, Jr., of the University of Oklahoma at Oklahoma City, Gary R. Cutter and Richard A. Windsor of the University of Alabama at Birmingham, and Patricia W. Wahl of the University of Washington.

I would like to thank the many students, instructors, and special reviewers for their thoughtful and helpful suggestions on improving this edition—specifically, K. M. Camarata, Eastern Kentucky University; Phyllis T. Croisant, Eastern Illinois University; James E. Hornak, Central Michigan University; and Paul E. Leaverton, University of South Florida. In particular, I am especially grateful to Steve Bohnenblust, Mankato State University, whose extensive work on this revision has been invaluable. I wish also to acknowledge the cooperation of the various publishers who generously granted permission to reproduce tables from their books.

J.W.K.

Statistics and How They Are Used

Chapter Outline

Learning Objectives

After studying this chapter, you should be able to

1. Define "statistics"

2. List several reasons for studying statistics

3. Distinguish clearly between
   a. descriptive and inferential statistics

b. surveys and experiments
c. retrospective and prospective studies
d. descriptive and analytical surveys

4. Define "bias"

5. Describe the purpose and components of a clinical trial

THE MEANING OF STATISTICS

One way to understand statistics is to consider two basic questions: What does the term *statistics* mean? What do statisticians do? Once we have the answers to these questions, we can delve into how statistics are used.

What Does "Statistics" Mean?

The word **statistics** has several meanings. It is frequently used to refer to recorded data such as the number of traffic accidents, the size of enrollment, or the number of patients visiting a clinic. Statistics is also used to denote characteristics calculated for a set of data—for example, mean, standard deviation, and correlation coefficient. In another context, statistics refers to statistical methodology and theory.

In short, statistics is a body of techniques and procedures dealing with the collection, organization, analysis, interpretation, and presentation of information that can be stated numerically.

What Do Statisticians Do?

A statistician is usually a member of a group that works on challenging scientific tasks. Frequently engaged in projects that explore the frontiers of human knowledge, the statistician is primarily concerned with developing and applying methods that can be used in collecting and analyzing data. He or she may select a well-established technique or develop a new one that may provide a unique approach to a particular study, thus leading to valid conclusions. Specifically, the statistician's tasks are as follows:

1. *To guide the design of an experiment or survey.* A statistician should be consulted in the early planning stages so that an investigation can be carried out efficiently, with a minimum of bias. Once data are collected, it is too late to plan ahead. By then, it is impossible to impose an appropriate statistical design or compensate for the lack of a randomly selected sample.

2. *To analyze data.* Data analysis may take many forms, such as examining the relationships among several variables, describing and analyzing the variation of certain characteristics (e.g., blood pressure, temperature, height, weight), or determining whether a difference in some response is significant.

3. *To present and interpret results.* Results are best evaluated in terms of probability statements that will facilitate the decision-making process. Mainland (1963:3) defines statistics as the "science and art of dealing with variation in such a way as to obtain reliable results." The art of statistics is especially pertinent to this task and involves skills usually acquired through experience.

Because the interpretation of statistics is more of an art than a science, it is all too easy to emphasize some inappropriate aspect of the results and consequently misuse statistics. An interesting little book, *How to Lie with Statistics* by Darrell Huff (1954), provides an enlightening and entertaining view of the problems involved in presenting statistics.

In accomplishing the previously-mentioned three tasks, a statistician generally reaches the major objective of statistics: to make an inference about a population being studied based on data collected from a sample drawn from this population.

THE USES OF STATISTICS

It is helpful to distinguish between the two major categories of statistics. **Descriptive statistics** deal with the enumeration, organization, and graphical representation of data. **Inferential statistics** are concerned with reaching conclusions from incomplete information—that is, generalizing from the specific. Inferential statistics use information obtained from a sample to say something about an entire population.

An example of *descriptive* statistics is the decennial **census** of the United States, in which all residents are requested to provide such information as age, sex, race, and marital status. The data obtained in such a census can then be compiled and arranged into tables and graphs that describe the characteristics of the population at a given time. An example of *inferential* statistics is an opinion poll, such as the Gallup Poll, which attempts to draw inferences as to the outcome of an election. In such a poll, a sample of individuals (frequently fewer than 2000) is selected, their preferences are tabulated, and inferences are made as to how more than 80 million persons would vote if an election were held that day.

Statistical methods provide a logical basis for making decisions in a variety of areas when incomplete information is available. Here are some examples of scientific questions to which the application of statistical methodology has been useful:

1. How can researchers test the effectiveness of a new vaccine against the common cold?

2. How effective is a trial that seeks to reduce the risk of coronary heart disease?

3. How effective have several family planning programs been?

4. How much, if at all, does use of oral contraceptives increase a woman's chances of developing a thromboembolism?

The three specific studies described next further amplify the application of statistics.

Smoking During Pregnancy A pioneering study of the effects on the newborn infant of smoking during pregnancy was reported by Simpson (1957). She examined the data of 7499 patients in three hospitals in and near Loma Linda University and found from the records that prematurity rates increased with the number of cigarettes smoked per day. A more recent review of the various studies on this topic is given by the Surgeon General's Report on Smoking and Health (U.S. Department of Health, Education, and Welfare, 1979). The principal conclusion of that report is: "Maternal smoking during pregnancy has a significant adverse effect upon the well-being of the fetus and the health of the newborn baby."

Health Practices and Mortality Belloc (1973) reported on a very interesting study conducted by the Human Population Laboratory of the California State Health Department on a representative sample of 6928 Alameda County residents. She concluded that there was a striking inverse relationship between the *number* of lifestyle practices (not smoking, not being obese, not drinking, being physically active, eating regularly) and mortality.

The Multiple Risk Factor Intervention Trial (MRFIT) Paul (1976) reported on a national study of the primary prevention of coronary heart disease. The study's approach was to determine whether the risk of coronary disease in middle-aged men can be significantly reduced through intervention. This intervention entailed simultaneously reducing their serum cholesterol levels, treating any high blood pressure, and encouraging the men to stop smoking. The seven-year trial involved 20 clinical centers and 12,866 subjects, all initially healthy but at high risk for coronary disease. At random, half the men were assigned to be followed through the intervention program and the other half through their usual medical care, which included annual physicals and lab tests. The report of the results was prepared by the MRFIT research group and appeared in the *Journal of the American Medical Association* (1982; 248:1465–1477). Investigators observed that the risk factor levels declined in both groups. Furthermore, during the seven-year follow-up period, the mortality rates for coronary heart disease (CHD) were 17.9 deaths per 1000 for the intervention group and 19.3 deaths per 1000 for the untreated group. This was a nonsignificant difference, and the lack of a positive result has generated considerable discussion. There may be more plausible reasons for this outcome: (1) it is difficult to show a significant drop due to an intervention when the entire country is experiencing a multidecade decline in CHD rates; (2) the intervention strategy may not have been drastic enough to show a significant difference; and (3) because a report of the assessed

risk factors was sent to the physicians of those in the untreated group, members of that group may have benefited from whatever "treatment" their physicians had prescribed for them.

Because skills, facilities, and funds are never unlimited, the problem arises as to how to extract the maximum amount of information in the most efficient manner. With the aid of statistics, it is usually possible to achieve greater precision at minimum cost by effectively using the resources available.

WHY STUDY STATISTICS?

Many students ask: "Why should I study statistics?" or "How useful will statistics be in my future career?" The answers to these questions depend on one's career objectives.

A knowledge of statistics is essential for people going into research management or graduate study in a specialized area. Persons active in research will find that a basic understanding of statistics is useful not only in the conduct of their investigations, but also in the effective presentation of their findings in papers, in reports for publication, and at professional meetings. Some proficiency in statistics is helpful to those who are preparing, or may be called upon to evaluate, research proposals. Further, a person with an understanding of statistics is better able to decide whether his or her professional colleagues use their statistics to illuminate or merely to support their personal biases; that is, it helps one to decide whether the claims are valid or not.

A knowledge of statistics is essential for persons who wish to keep their education up-to-date. To keep abreast of current developments in one's field, it is important to review and understand the writings in scientific journals, many of which use statistical terminology and methodology.

An understanding of statistics can help anyone discriminate between fact and fancy in everyday life—in reading newspapers and watching television, and in making daily comparisons and evaluations.

Finally, a course in statistics should help one know when, and for what purpose, a statistician should be consulted.

SOURCES OF DATA

In observing various phenomena, we are usually interested in obtaining information on specific characteristics—for instance, age, weight, height, marital status, or smoking habits. These characteristics are referred to as **variables;** the values of the observations recorded for them are referred to as **data.** Data are the raw materials of statistics. They are derived from incredibly diverse sources. Knowing our sources provides clues to our data—their reliability, their validity, and the inferences we might draw.

Surveys and Experiments

Data may come from anywhere: observational surveys, planned surveys, or experiments. The two fundamental kinds of investigations are **surveys** and **experiments.** Data from a survey may represent observations of events or phenomena over which few, if any, controls are imposed. The study of the effects of the explosion of the atomic bomb on the inhabitants of Hiroshima and Nagasaki is an example of a survey. In this case, the radiation to which the survivors were exposed (referred to as "treatment" in statistics) was in no way controlled or assigned. By contrast, in an experiment, we design a research plan purposely to impose controls over the amount of exposure (treatment) to a phenomenon such as radiation. The distinction between them is that an experiment imposes controls on the methods, treatment, or conditions under which it is performed, whereas in a survey such controls are seldom possible.

A classic example of an experiment is the Veterans Administration Cooperative Study. It began in 1963 and involved 523 hypertensive men who were patients in 16 Veterans Administration hospitals (Veterans Administration, 1970, 1972). The study demonstrated that oral hypertensive medications, judiciously administered, could significantly reduce blood pressure levels, whereas **placebos** (substances or treatments that have no therapeutic value) had no effect on blood pressure.

Although experimental investigations are preferable to surveys, in some cases there are reasons for not conducting them—for instance, ethical reasons, as when a beneficial treatment may be withheld from one of the groups; or administrative reasons, as when an experiment may seriously disrupt the established routine of patients' care.

Health researchers conduct surveys on human populations all the time. These surveys may be categorized as retrospective or prospective.

Retrospective Studies

Retrospective studies (commonly referred to as **case-control studies**) gather past data from selected cases and controls to determine differences, if any, in the exposure to a suspected factor. In retrospective studies, the researcher identifies individuals with a specific disease or condition (cases) and also identifies a comparable sample without that disease or condition (controls). The purpose of the comparison is to determine if the two groups differ as to their exposure to some specific factor. An example is a study that compares the smoking habits of women who bore premature babies to those of women who carried their pregnancies to term. Given the comparative data, the researcher then seeks to determine whether there is a statistical relation between the possible **stimulus variable,** or causative factor (smoking), and the **outcome variable** (prematurity).

A disadvantage of retrospective studies is that the data were usually collected for other purposes and may be incomplete. Surveys frequently fail to include relevant variables that may be essential to determine whether the two

Table 1.1 Generalized 2 × 2 Table

Stimulus Variable	Outcome Variable		Total
	With Disease	Without Disease	
Present	a	b	$a + b$
Absent	c	d	$c + d$
Total	$a + c$	$b + d$	

groups studied are comparable. This absence of demonstrated comparability between cases and controls may envelop the results in a cloud of doubt. In addition, because of the historical nature of such records or the necessity of relying on memory, serious difficulties may attend the selection of appropriate controls. Unknown biases frequently hinder such studies.

The major advantages of retrospective studies are that they are economical and are particularly applicable to the study of rare diseases. Such studies also make it possible to obtain answers relatively quickly because the cases are usually easily identified.

In retrospective studies, sample selection begins with the outcome variable (disease). The researcher looks back in time to identify the stimulus variable (factor). In prospective studies (discussed next), the stimulus variable is known in advance and the study population is followed through time, while occurrences of the outcome are noted. A generalized **2 × 2 table** may be used to illustrate the study design (Table 1.1). This table is applicable to both retrospective and prospective studies and is called a fourfold table because it consists of four elements, a, b, c, and d:

Element a represents persons with the stimulus variable who developed the disease.

Element b represents persons with the stimulus variable who did not develop the disease.

Element c represents persons without the stimulus variable who developed the disease.

Element d represents persons without the stimulus variable who did not develop the disease.

Prospective Studies

Prospective studies are usually **cohort studies,** in which one enrolls a group of healthy persons and follows them over a certain period to determine the frequency with which a disease develops. The group is divided statistically

according to the presence or absence of a stimulus variable (e.g., smoking history). This is done because the group cannot, of course, be divided according to a disease that has not yet occurred (e.g., the presence or absence of lung cancer). The prospective study then compares the proportion of smokers (exposed cohort) who developed lung cancer to the proportion of nonsmokers (nonexposed cohort) who developed the same disease.

The prime advantage of prospective studies is that they permit the accurate estimation of disease incidence in a population. They make it possible to include relevant variables, such as age, sex, and occupation, that may be related to the outcome variable. Furthermore, they permit data collection under uniform conditions: Data are obtained for specified reasons and there are better opportunities to make appropriate comparisons while limiting or controlling the amount of **bias,** which may be considered *systematic error.* The disadvantages of prospective studies are that they take considerable time and are expensive in studying diseases of low incidence.

A good example of a prospective study is one that seeks to determine if there are long-term health effects on women who take oral contraceptives. Prospective studies do not prove a causal relationship with the factor under study because the characteristics (such as smoking or not smoking) are not randomly assigned and persons with an inherent tendency to lung cancer are arguably more likely to be included in the smoking group. Nevertheless, such studies provide the best mechanism for providing "causal" evidence. The results should be taken as important—though less than perfect—scientific evidence. In some studies, such as those of smoking and lung cancer, the relationship, although not proven, may well be established beyond a reasonable doubt. On this point, MacMahon and Pugh (1970:22) aptly state, "When the derivation of experiential evidence is either impracticable or unethical, there comes a point in the accumulation of evidence when it is more prudent to act on the basis that the association is causal rather than to await further evidence."

Comparison of Ratios

For each type of study, it is instructive to note the different ratios that can be constructed and the questions that can be answered. For *retrospective* studies the ratios to be compared (using the notation of Table 1.1) are

$$\frac{a}{a + c} \quad \text{and} \quad \frac{b}{b + d}$$

By comparing them we can answer the question: Were mothers of premature infants more likely to have been smokers than mothers of normal infants?

For *prospective* studies the ratios to be compared are

$$\frac{a}{a + b} \quad \text{and} \quad \frac{c}{c + d}$$

This comparison answers the question: Which group has the higher frequency of premature infants—mothers who smoke or mothers who do not smoke?

Descriptive and Analytical Surveys

Retrospective surveys are usually **descriptive.** Such surveys provide estimates of a population's characteristics, such as the proportion of individuals who had a physical examination during the last 12 months. Prospective surveys may be descriptive or **analytical.** In an analytical survey one seeks to determine the degree of association between a variable and a factor in the population. An example is the relationship between having (or not having) regular physical examinations and some measure of health status.

CLINICAL TRIALS

A **clinical trial** is a carefully designed experiment that seeks to determine, under controlled conditions, the effectiveness of a new drug or treatment method. Clinical trials are used extensively today by investigators seeking to determine the effectiveness of newly proposed drugs, such as cancer chemotherapeutic agents. One of the pioneer clinical trials evaluated the effectiveness of streptomycin in the treatment of tuberculosis (Medical Research Council, 1948). Other clinical trials have been used to evaluate polio vaccine, ACTH for multiple sclerosis, tolbutamide for the control of diabetes, and hundreds of new cancer chemotherapeutic agents.

In short, a clinical trial involves a comparison of two or more comparable groups of patients. The **treatment group,** which receives a potentially therapeutic agent, is compared with a similar **control group,** which instead receives a placebo or the standard therapeutic treatment. It is important that the two groups of patients be comparable. To ensure that they are, patients are usually **randomly allocated**—that is, each patient is given an equal chance of being assigned to the treatment or the control group.

The investigator is interested not only in establishing comparable groups, but also in limiting the amount of bias entering a trial. One way to do this is to design the experiment as a **single-blind study.** In this type of experiment, the patient does not know whether he or she is in the treatment or the control group. An even better way is to design it as a **double-blind study.** Here, neither the patient nor the experimenter knows to which group the patient is assigned. A neutral party keeps the code as to who's who and discloses it only at the end of data gathering. Numerous clinical trials have failed because bias was not adequately controlled. Bias falls into a number of categories, and is discussed further in Chapter 17.

A clinical trial demands an appropriate control group. One such group is a *concurrent* control group, which is selected at the same time and from the same pool of individuals as the treatment group. Because the use of controls at least

doubles the size of the experiment, some investigators have tried alternatives, such as historical controls. Historical controls are control subjects that are selected from a set of records after a study has been completed. But historical controls present problems because of changes in the population over time and because there is no way to control selection bias. Volunteer groups have also been used as controls. Because such a group is self-selected, however, it is usually atypical of the rest of the population, thus limiting the inferences that may be drawn. Some investigators have chosen controls from patients in certain hospital wards. This method presents problems of selection for a particular kind of disease or condition. It may overrepresent patients hospitalized for a long time or those recently admitted. Because a clinical trial is, in actuality, an experiment on human beings, a number of ethical issues also arise. For instance, is it ethical to withhold a probably effective mode of treatment from a control group? For further discussion of such problems see Hill (1963) and Colton (1974). For a step-by-step procedure of preparing a protocol for a clinical trial, see Kuzma (1970).

Clinical trials as used today have developed since World War II and are extremely helpful in distinguishing between effective and ineffective agents. Had clinical trials been used more commonly in the early days of medicine, the futility of such drastic and dangerous methods as bloodletting and purging would have been exposed early on.

In summary, then, the salient features of a clinical trial are

1. Simultaneous treatment and control groups
2. Subjects who are randomly allocated to the two groups
3. Use of a double-blind technique when feasible

Example: The Salk Vaccine Clinical Trial

The 1954 clinical trial of the Salk poliomyelitis vaccine is a good example of how a clinical trial can be used to solve an important public health problem. At that time, outbreaks of polio were unpredictable. Because the disease caused paralysis and frequently death, such outbreaks were of great concern to both parents and children. Enter Dr. Jonas Salk. Salk developed a vaccine that proved safe and effective in a laboratory setting in producing antibodies against polio. The question to be answered then was whether this promising vaccine could prevent polio in exposed individuals.

To find the answer, a clinical trial was set up. Statisticians recommended that at least 400,000 children be included in the study: 200,000 children in the treatment group and 200,000 children in the control group. The large numbers were needed to provide an adequate number of cases in order to get valid results. An adequate number could be obtained only with these large sample sizes because the incidence rate of polio cases was estimated to be 35 per 100,000 children. The 400,000 children in the study were randomly assigned to either a treatment

group (the group receiving the active Salk vaccine) or to a control group (the group receiving a placebo, which consisted of an injection of salt dissolved in water). Because of this precaution—the addition of the double-blind feature— neither the children nor the administrators of the treatment knew which child received the vaccine and which received the placebo. Furthermore, those who examined the children to determine whether they had contracted polio were also unaware of their patients' group status.

It was important that the study group be randomly allocated so that the two groups would be comparable. If this procedure had not been followed, it is likely that the treatment group would have been biased because children from higher socioeconomic levels, whose parents were aware that they were at greater risk, would more likely have participated. Such children were at greater risk because their environment was more hygienic than that of children from lower socioeconomic strata, and they were less likely to have developed an immunity to the disease.

The tabulation of the collected data indicated that the incidence rate of cases in the treatment group was 28 per 100,000 versus 71 per 100,000 in the control group. Statistical analysis of these rates showed that the Salk polio vaccine was indeed effective, and that the clinical trial (one of the largest ever) and its cost ($5 million) were justified.

Some students may be concerned about the ethical problem of withholding treatment from half of the study group. However, before the clinical trial, there was no definite proof of the effectiveness of the Salk polio vaccine, and, without a control group, there was no available scientific, rigorous procedure by which to provide definitive answers. A clinical trial had to be carried out. Once it was—and the evidence was convincing—the public health authorities had the evidence necessary to mount a national campaign to virtually eradicate polio. Their efforts were successful—in 1952 there were about 60,000 cases of polio; today there are hardly any (Thomas, 1955).

PLANNING OF SURVEYS

The previous section discussed several types of medical surveys that may give rise to data. Before starting a survey, it is essential to formulate a clear plan of action. An outline of such a plan, including the major steps that should be followed in pursuing the investigation, is given in Chapter 17.

HOW TO SUCCEED IN STATISTICS

Studying statistics is somewhat analogous to studying a foreign language because a considerable number of new terms and concepts need to be learned. We have found that students who do this successfully scan the chapter outline, read

the conclusion and the vocabulary list, and review the learning objectives before coming to class. Also, as soon as possible after the class, they study and learn the relevant terms, concepts, principles, and formulae in the textbook. After doing the assigned exercises, they try to reformulate the objectives as questions and then answer them. We suggest that you do the same. The questions you form from the objectives also serve as excellent review questions you can use to help prepare for examinations. If you are not sure of some of these objectives, you may need to go back and reread the chapter or do additional exercises. Doing as many exercises as possible is one of the best ways to learn statistics. If anything is still not clear, make up questions you can ask at the tutorial session or in class.

In addition, read essays dealing with the application of statistics to a variety of fields. An excellent and readable book is *Statistics: A Guide to the Unknown* by Judy M. Tanur et al. (1978). Also, because many of the exercises involve a large number of measurements, you may find a calculator helpful. Finally, keep in mind that students who are successful in mastering statistics do not allow themselves to get behind.

Conclusion

A statistician designs efficient and unbiased investigations that provide data that he or she then analyzes, interprets, and presents to others so that decisions can be made. To do this work, the statistician uses techniques that are collectively called statistics. Students of statistics learn these techniques and how they may relate to their work and everyday life. Particularly they learn how to make correct inferences about a target population of interest based on sample data. Students need to know not only how to understand the scientific literature of their field but also how to select from various kinds of investigations the one that best fits their research purpose.

Vocabulary List

analytical survey	descriptive survey	single-blind study
bias	double-blind study	statistics
case-control study	experiment	stimulus variable
census	inferential statistics	survey
clinical trial	outcome variable	treatment group
cohort study	placebo	two-by-two table
control group	prospective study	(2 × 2 table)
data	random allocation	variable
descriptive statistics	retrospective study	

Exercises

1.1 Suggest and describe briefly a survey and its objectives.
a. Is it a descriptive or an analytical survey?
b. List some potential sources of bias.

1.2 Suggest and describe an experiment.
a. What research question are you testing?
b. What is the "treatment" in this experiment?
c. List some potential sources of bias.

1.3 Suggest a clinical trial for some phenomenon of interest to you, such as drug use or exercise.
a. Describe how you would select and allocate cases.
b. What would be the treatment?
c. What would be the outcome variable for determining the effectiveness of the treatment?
d. What double-blind feature would you include, if any?

1.4 Find a newspaper or magazine article that uses data or statistics.
a. Were the data obtained from a survey or an experiment?
b. Is the study descriptive or inferential?
c. What research question was the author trying to answer?
d. How did he or she select the cases? What population do the cases represent?
e. Was there a control group? How were the control subjects selected?
f. Are possible sources of bias mentioned?
g. If conclusions are stated, are they warranted?
h. Make a copy of the article to turn in with your answers to these questions.

1.5 Define: bias, clinical trial, experiment, survey, and statistics.

1.6 Explain what is meant by
a. descriptive statistics
b. inferential statistics

1.7 Answer the following questions regarding the Salk vaccine trial:
a. Why was such a large trial necessary?
b. Why was a control group needed?
c. Why is it important to include a double-blind feature?
d. If volunteers were used in this trial rather than a random sample of individuals, of what value would be the results?

1.8 U.S. census statistics show that college graduates make more than $254,000 more in their lifetime than non-college graduates. If you were to question the validity of this observation, what would be your basis for doing so?

Populations and Samples

Chapter Outline

2.1 Selecting Appropriate Samples
Explains why the selection of an appropriate sample has an important bearing on the reliability of inferences made about a population

2.2 Why Sample?
Gives a number of reasons as to why sampling is often preferable to census taking

2.3 How Samples Are Selected
Shows several ways in which samples are selected

2.4 How to Select a Random Sample
Illustrates with a specific example the method of selecting a sample by the use of a random number table

2.5 Effectiveness of a Random Sample
Demonstrates the credibility of the random sampling process

Learning Objectives

After studying this chapter, you should be able to

1. Distinguish between
   a. population and sample
   b. parameter and statistic
   c. the various methods of sampling

2. Explain why the method of selecting a sample is important

3. State the reasons why samples are used

4. Define a random sample

5. Explain why it is important to use random sampling

6. Select a sample using a random number table

SELECTING APPROPRIATE SAMPLES

A **population** is a set of persons (or objects) having a common observable characteristic. A **sample** is a subset of a population.

The real challenge of statistics is how to come up with a reliable statement about a population on the basis of sample information. For example, if we want to know how many persons in a community have quit smoking or have health insurance or plan to vote for a certain candidate, we usually obtain information on an appropriate sample of the community and generalize from it to the entire population. How a subgroup is selected is of critical importance. Take the classic example of the *Literary Digest Poll*. The Literary Digest Poll attained considerable prestige by successfully predicting the outcomes of four presidential elections before 1936. Using the same methods, the *Literary Digest* in 1936 mailed out some 10 million ballots asking persons to indicate their preference in the upcoming presidential election. About 2.3 million ballots were returned, and based on these, the *Literary Digest* confidently predicted that Alfred M. Landon would win by a landslide. In fact, Franklin D. Roosevelt won with a 62% majority. Soon after this fiasco the *Literary Digest* ceased publication. A postmortem examination of its methods revealed that the sample of 10 million was selected primarily from telephone directories and motor vehicle registration lists, which meant that the poll was overrepresented by persons with high incomes. In 1936 there was a strong relation between income and party preference; thus, the poll's failure was virtually inevitable.

The moral of this incident is clear. The *way* the sample is selected, not its *size,* determines whether we may draw appropriate inferences about a population. Modern sampling techniques can quite reliably predict the winner of a presidential election from a nationwide sample of less than 2000 persons. This is remarkable, considering that the nation's population today is more than twice as large as it was in 1936.

Here are some examples of populations that one may wish to sample: veterans of foreign wars, marijuana users, persons convicted of driving while intoxicated, persons who have difficulty gaining access to medical care, gifted children, or residents of a certain city. The primary reason for selecting a sample from a population is to draw inferences about that population. Note that the population may consist of persons, objects, or the observations of a characteristic. The set of observations may be summarized by a descriptive characteristic, called a **parameter.** When the same characteristic pertains to a sample, it is called a **statistic.** Sample statistics help us draw inferences about population parameters.

The value of the population parameter is constant but usually unknown. The value of the statistic is known because it is computed from the sample. Observations differ from one sample to the next; consequently, the value of the statistic varies from sample to sample.

WHY SAMPLE?

You may be wondering, "Why not study the entire population?" There are many reasons. It is *impossible* to obtain the weight of every tuna in the Pacific Ocean. It is too costly to inspect every manifold housing that comes off an assembly line. The Internal Revenue Service does not have the workforce to review every income tax return. Some testing is inherently destructive: tensile strength of structural steel, flight of a solid propellant rocket, measurement of white blood count. We certainly cannot launch all the rockets to learn the number of defective ones; we cannot drain all the blood from a person and count every white cell. Often we cannot justify enumerating the entire population—that is, conducting a census—because for most purposes we can obtain suitable accuracy quickly and inexpensively on the basis of the information gained from a sample alone. One of the tasks of a statistician is to design efficient studies utilizing adequate sample sizes that are not unnecessarily large. How to determine a sample size that is likely to give meaningful results is discussed in Chapter 8.

HOW SAMPLES ARE SELECTED

How reliable are our inferences regarding a population? The answer to this depends on how well the population is specified and on the method of sample selection. Having a poorly specified or enumerated population or an inappropriately selected sample will surely introduce bias. But bias is controllable. The best way to limit bias is to use **random sampling,** a technique that is simple to apply (which is why it is sometimes called *simple random sampling*). We use a means of randomization such as a random number table (described in the next section) to ensure that each individual in the population has an equal chance of being selected. This technique meets some of the important assumptions underlying several statistical methods. It also makes possible the estimation of error.

Samples can be selected in several other ways. In **convenience sampling,** a group is selected at will or in a particular program or clinic. These cases are often self-selected. Because the data obtained are seldom representative of the underlying population, problems arise in analysis and in drawing inferences.

Convenience samples are often used when it is virtually impossible to select a random sample. For instance, if a researcher wants to study alcohol use among college students, ideally each member of the population—that is, each college student—would have an equal chance of being sampled. A random sample of 100, 1000, or 10,000 college students is simply not realistic. How will the researcher collect data about alcohol use among college students? Often the researcher will survey college students enrolled in a general education course, such as English 101. The underlying assumption on the part of the researcher is that a general education class, which most or all students must take, is a representative sample of college students and therefore accurately represents alcohol

use at that college or university. The logical next step is to assume that the colleges or universities surveyed are representative in terms of college and university students' alcohol use. One can see how the use of a convenience sample may eventually lead researchers to inferences about alcohol use among college students that are inaccurate or misleading.

Systematic sampling is frequently used when a **sampling frame** (a complete, nonoverlapping list of the persons or objects constituting the population) is available. We randomly select a first case and then proceed by selecting every nth (say $n = 30$) case, where n depends on the desired sample size. The symbol N is used to denote the size of the entire population.

Stratified sampling is used when we wish the sample to represent the various **strata** (subgroups) of the population proportionately or to increase the precision of the estimate. A simple random sample is taken from each stratum.

In **cluster sampling,** we select a simple random sample of groups, such as a certain number of city blocks, and then interview a person in each household of the selected blocks. This technique is more economical than the random selection of persons throughout the city.

For a complete discussion of the various kinds of sampling methods, you should consult a textbook on the subject. A good one is by Scheaffer, Mendenhall, and Ott (1979).

HOW TO SELECT A RANDOM SAMPLE

One of the easiest ways to select a random sample is to use a **random number table.** Such tables are easy to find; they are in many statistical texts and mathematical handbooks. Many calculators also generate random numbers. A portion of a random number table is reproduced in Table 2.1, and an additional random number table is included in Appendix E. Random number tables are prepared in such a way that each digit gets equal representation. Selecting a random sample involves three steps: (1) Define the population, (2) enumerate it, and (3) use a random number table to select the sample.

Here is an illustration of how to select 10 persons from a population of 83 cases in a hypertension study (see Table 2.2). Observe that the population is clearly defined: 83 cases classified according to their diastolic blood pressure, sex, and dietary status. Also note that the cases have been numbered arbitrarily from 01 to 83. If the random number table covered four pages, you might flip a coin twice and arbitrarily agree to assign HH (two heads) to page 1, HT to page 2, TH to page 3, and TT to page 4. Suppose you flip heads on both tosses (HH); you then turn to page 1. To choose an arbitrary starting place, you could blindly stab at row 19 and column 31. (This procedure is illustrated in Table 2.1.) The row–column intersection of the starting place should be recorded, just in case you wish later to verify your selection and hence your sample. Next, read the two-digit numeral that falls at that spot. Two digits are used because your sampling frame is identified by two-digit numerals. The first number selected is 24.

Table 2.1 Random Numbers

					col. 31 ↓					
00439	81846	45446	93971	84217	74968	62758	49813	13666	12981	
29676	37909	95673	66757	72420	40567	81119	87494	85471	81520	
69386	71708	88608	67251	22512	00169	58624	04059	05557	73345	
68381	61725	49122	75836	15368	52551	54604	61136	51996	19921	
69158	38683	41374	17028	09304	10834	61546	33503	84277	44800	
00858	04352	17833	41105	46569	90109	14713	15905	84555	92326	
86972	51707	58242	16035	94887	83510	56462	83759	68279	64873	
30606	45225	30161	07973	03034	82983	78242	06519	96345	53424	
93864	49044	57169	43125	11703	87009	76463	48263	99273	79449	
61937	90217	56708	35351	60820	90729	90472	68749	23171	67640	
94551	69538	52924	08530	79302	34981	12155	42714	39810	92772	
79385	49498	48569	57888	70564	17660	50411	19640	07597	34550	
14796	51195	69638	55111	06883	13761	53688	44212	71380	56294	
79793	05845	58100	24112	26866	26299	74127	63514	04218	07584	
98488	68394	65390	41384	52188	81868	74272	77608	34806	46529	
96773	24159	28290	31915	30365	06082	73440	16701	78019	49144	
18849	96248	46509	56863	27018	64818	40938	66102	65833	39169	
71447	27337	62158	25679	63325	98669	16926	28929	06692	05049	
97091	42397	08406	04213	52727	08328 →	24057	78695	91207	18451	← **row**
56644	52133	55069	57102	67821	54934	66318	35153	36755	88011	**19**
60138	40435	75526	35949	84558	13211	29579	30084	47671	44720	
80089	48271	45519	64328	48167	14794	07440	53407	32341	30360	
54302	81734	15723	10921	20123	02787	97407	02481	69785	58025	
61763	77188	54997	28352	57192	22751	82470	92971	29091	35441	
25769	28265	26135	52688	11867	05398	43797	45228	28086	84568	
80142	64567	38915	40716	76797	37083	53872	30022	43767	60257	
69481	57748	93003	99900	25413	64661	17132	53464	52705	69602	
40431	28106	28655	84536	71208	47599	36136	46412	99748	76167	
16264	39564	37178	61382	51274	89407	11283	77207	90547	50981	
19618	87653	18682	22917	56801	81679	93285	68284	11203	47990	

By advance agreement, you could proceed by reading down the column: 66, 29, 7, 97, and so on. Alternatively, you could agree to read the table in some other reasonable way, say from left to right. Whatever the pattern you choose, you cannot change it during the selection process. Continuing to read down the columns, you would select the following individuals:

ID	Diastolic Blood Pressure
24	58
66	82
29	56
7	58
82	66
43	102
53	92
17	68
36	60
11	78

Table 2.2 Hypertension Study Cases by Diastolic Blood Pressure, Sex, and Dietary Status

ID	Diastolic Blood Pressure (mmHg)	Sex	Vegetarian Status	ID	Diastolic Blood Pressure (mmHg)	Sex	Vegetarian Status
01	88	M	V	42	70	M	NV
02	98	M	V	43	102	M	NV
03	64	M	V	44	84	M	NV
04	80	M	V	45	74	M	NV
05	60	M	V	46	76	M	NV
06	68	M	V	47	84	M	NV
07	58	M	V	48	84	M	NV
08	82	M	V	49	82	M	NV
09	74	M	V	50	82	M	NV
10	64	M	V	51	74	M	NV
11	78	M	V	52	70	M	NV
12	68	M	V	53	92	M	NV
13	60	M	V	54	68	M	NV
14	96	M	V	55	70	M	NV
15	64	M	V	56	70	M	NV
16	78	M	V	57	70	M	NV
17	68	M	V	58	40	M	NV
18	72	M	V	59	83	M	NV
19	76	F	V	60	74	M	NV
20	68	F	V	61	56	F	NV
21	70	F	V	62	89	F	NV
22	62	F	V	63	84	F	NV
23	82	F	V	64	58	F	NV
24	58	F	V	65	58	F	NV
25	72	F	V	66	82	F	NV
26	56	F	V	67	78	F	NV
27	84	F	V	68	82	F	NV
28	80	F	V	69	71	F	NV
29	56	F	V	70	56	F	NV
30	58	F	V	71	68	F	NV
31	82	F	V	72	58	F	NV
32	88	F	V	73	72	F	NV
33	100	F	V	74	80	F	NV
34	88	F	V	75	88	F	NV
35	74	F	V	76	72	F	NV
36	60	F	V	77	68	F	NV
37	74	F	V	78	66	F	NV
38	70	F	V	79	78	F	NV
39	70	F	V	80	74	F	NV
40	66	F	V	81	60	F	NV
41	76	M	NV	82	66	F	NV
				83	72	F	NV

NOTE: ID = identification; mmHg = millimeters of mercury; V = vegetarian; NV = nonvegetarian.

Table 2.2 Hypertension Study Cases by Diastolic Blood Pressure, Sex, and Dietary Status

ID	Diastolic Blood Pressure (mmHg)	Sex	Vegetarian Status	ID	Diastolic Blood Pressure (mmHg)	Sex	Vegetarian Status
01	88	M	V	42	70	M	NV
02	98	M	V	43	102	M	NV
03	64	M	V	44	84	M	NV
04	80	M	V	45	74	M	NV
05	60	M	V	46	76	M	NV
06	68	M	V	47	84	M	NV
07	58	M	V	48	84	M	NV
08	82	M	V	49	82	M	NV
09	74	M	V	50	82	M	NV
10	64	M	V	51	74	M	NV
11	78	M	V	52	70	M	NV
12	68	M	V	53	92	M	NV
13	60	M	V	54	68	M	NV
14	96	M	V	55	70	M	NV
15	64	M	V	56	70	M	NV
16	78	M	V	57	70	M	NV
17	68	M	V	58	40	M	NV
18	72	M	V	59	83	M	NV
19	76	F	V	60	74	M	NV
20	68	F	V	61	56	F	NV
21	70	F	V	62	89	F	NV
22	62	F	V	63	84	F	NV
23	82	F	V	64	58	F	NV
24	58	F	V	65	58	F	NV
25	72	F	V	66	82	F	NV
26	56	F	V	67	78	F	NV
27	84	F	V	68	82	F	NV
28	80	F	V	69	71	F	NV
29	56	F	V	70	56	F	NV
30	58	F	V	71	68	F	NV
31	82	F	V	72	58	F	NV
32	88	F	V	73	72	F	NV
33	100	F	V	74	80	F	NV
34	88	F	V	75	88	F	NV
35	74	F	V	76	72	F	NV
36	60	F	V	77	68	F	NV
37	74	F	V	78	66	F	NV
38	70	F	V	79	78	F	NV
39	70	F	V	80	74	F	NV
40	66	F	V	81	60	F	NV
41	76	M	NV	82	66	F	NV
				83	72	F	NV

NOTE: ID = identification; mmHg = millimeters of mercury; V = vegetarian; NV = nonvegetarian.

Why was number 97 excluded? The answer is simple: The sampling frame defines only numbers 01 through 83. Disregard all others. A corollary problem is the duplication of a number already selected; in practice, the duplicate is simply ignored. It is important to remember that the selected numerals only identify the sample; the sample itself is the set of blood pressure values.

Occasionally it may be uneconomical or impractical to implement a random selection scheme that requires enumeration of the entire population. For example, it would be nearly impossible to obtain a list of all persons in a city who have a sexually transmitted disease or are obese. In such cases one might have to resort to whatever lists are conveniently available:

Simple random sampling is well named—you can see how simple it is to apply. Yet it is one of the statistician's most vital tools and is used in countless applications. It is the basic building block for every method of sampling, no matter how sophisticated.

2.5   EFFECTIVENESS OF A RANDOM SAMPLE

Students who encounter random sampling for the first time are somewhat skeptical about its effectiveness. The reliability of sampling is usually demonstrated by defining a fairly small population and then selecting from it all conceivable samples of a particular size, say three observations. Then, for each sample, the mean (average) is computed and the variation from the population mean is observed. A comparison of these sample means (statistics) with the population mean (parameter) neatly demonstrates the credibility of the sampling scheme.

In this chapter we try to establish credibility by a different approach. If you look ahead to Chapter 3, you will find Table 3.1, which lists characteristics of a representative sample of 100 individuals from the 7683 participants in the Honolulu Heart Study, which investigated heart disease among men ages 45 through 67. Five separate samples of 100 observations each were selected from this population, and the mean ages were compared with the population mean. The results of this comparison are shown in Table 2.3. We can see that the population parameter is 54.36 and that the five statistics representing this mean are all very close to it. The difference between the sample estimate and the population mean never exceeds 0.5 years, even though each sample represents only 1.3% of the entire population. This comparison underscores how much similarity you can expect among sample means.

Table 2.3 Effectiveness of a Random Sample

	Population $(N = 7683)$	Sample ($n = 100$ each)				
		1	2	3	4	5
Mean age	54.36	54.85	54.31	54.31	54.67	54.02

Conclusion

Assessing all individuals may be impossible, impractical, expensive, or inaccurate, so it is usually advantageous to study instead a sample of the original population. To do this we must clearly identify the population, be able to list it in a sampling frame, and utilize an appropriate sampling technique. Although several methods of selecting samples are possible, random sampling is the most practical in that it is easy to apply, limits bias, provides estimates of error, and meets the assumptions of the statistical tests. The effectiveness of random sampling can easily be demonstrated by comparing sample statistics with population parameters. The statistics obtained from a sample are used as estimates of the unknown parameters of the population.

Vocabulary List

cluster sample	random number table	statistic
convenience sample	random sample	stratified sample
parameter	sample	stratum (pl. strata)
population	sampling frame	systematic sample

Exercises

2.1 Draw a sample of 10 from Table 2.2 by the use of the random number table (Table 2.1). Make note of
   a. the row and column where you started
   b. the direction in which you proceeded
   c. the 10 values you selected (show the ID and blood pressure for each)
   What is this type of sample called?

2.2 Suppose in Exercise 2.1 you had selected the sample by taking two simple random samples of five from each of the two diet groups. What name would you apply to such a sample?

2.3 Select a sample of 10 from Table 2.2 by taking every eighth individual beginning with ID number 6.
   a. What is the name of such a sample?
   b. Do you see a possible source of bias in taking the sample in this way?

2.4 Look ahead to the blood glucose values listed in Table 3.1.
   a. Describe the population.
   b. Select a simple random sample of 10.
   c. What statistical term describes the characteristic for the sample?
   d. What statistical term describes the characteristic for the population?

2.5 Describe the differences between
   a. a parameter and a statistic
   b. a sample and a census
   c. a simple random sample and a convenience sample

2.6 Why is the way a sample is selected more important than the size of the sample?

2.7 Describe the population and sample for
a. Exercise 2.1
b. the data in Table 3.1

2.8 Describe the steps you would take if you were asked to determine
a. the proportion of joggers in your community
b. the number of workers without health insurance at companies with fewer than 100 workers in your community
c. the number of pregnant women not obtaining prenatal care in your community
d. the number of homeless people in your community

2.9 a. In what ways are a random sample, convenience sample, and systematic sample different? In what ways are they similar?
b. Describe a situation in which it would be appropriate and more convenient to use a random sample, a convenience sample, a systematic sample, or a cluster sample.

2.10 a. Explain why a convenience sample, such as the selection of students in one or more general education classes at your college or university, may not be representative of students at your institution.
b. Explain why students at your college or university may not be representative of students in general.

Organizing and Displaying Data

Chapter Outline

3.1   The Use of Numbers in Organizing Data
Discusses the three types of numbers in relation to organizing data

3.2   Quantitative and Qualitative Data
Draws a distinction among qualitative data, discrete quantitative data, and continuous quantitative data

3.3   The Frequency Table
Gives instructions on how to organize data in the form of a frequency table

3.4   Graphing Data
Discusses and illustrates various methods of graphing, with emphasis on those that apply specifically to frequency distributions

Learning Objectives

After studying this chapter, you should be able to

1. Distinguish between
   a. qualitative and quantitative variables
   b. discrete and continuous variables
   c. symmetrical, bimodal, and skewed distributions
   d. positively and negatively skewed distributions

2. Construct a frequency table that includes class limits, class frequency, relative frequency, and cumulative frequency

3. Indicate the appropriate types of graphs that can be used for displaying quantitative and qualitative data

4. Distinguish which form of data presentation is appropriate for different situations

5. Construct a histogram, a frequency polygon, an ogive, a bar chart, and a box and whisker plot

6. Interpret a frequency table

7. Distinguish among and interpret the various kinds of graphs

8. Determine and interpret percentiles from an ogive

3.1   THE USE OF NUMBERS IN ORGANIZING DATA

There are three general ways of organizing and presenting data: tables, graphs, and numerical methods. Each of these ways will be illustrated by reference to a sample of 100 individuals, selected by systematic random sampling from a Honolulu Heart Study population of 7683 (Phillips, 1972). The data for this sample are presented in Table 3.1.

First, however, we must take a few moments to discuss the subject of *numbers*. There are many different types of numbers. Those used most frequently in everyday life—telephone, zip code, social security, driver's license, and the like—do not represent an amount or quantity. Such numbers are used as names or identifiers of a person's status, category, or attribute and are referred to as **nominal numbers.** In Table 3.1, the nominal variables are the ID number and smoking status (smoker vs. nonsmoker).

An **ordinal** is another kind of number. Ordinals represent an ordered series of relationships. *First, second,* and *third* are ordinals. They may be applied, for example, to the rank order of causes of death by type of disease. Note that an ordinal indicates position in an ordered series but says nothing at all about the magnitude of difference between any two successive entries. In Table 3.1, educational level and physical activity status are examples of ordinal variables.

A third kind of number is one measured on an interval scale. An **interval scale** has equal units but an arbitrary zero point. Temperature is an example of an interval scale datum. Interval scale units may be added or subtracted but they may not be multiplied or divided. Common statistics such as the mean, standard deviation, and t can be computed on interval scale data. For example, an average age or height is meaningful, whereas an average zip code (a nominal variable) is senseless. It is *not* appropriate to perform arithmetic operations on nominal data. Variables such as weight (or height), which we can compare meaningfully with one another (say, 50 kg is twice 25 kg), are said to be measured on a **ratio scale.**

3.2   QUANTITATIVE AND QUALITATIVE DATA

As noted in Chapter 1, specific characteristics (e.g., age, height, and weight) that we may want to assess for a certain population are referred to as variables. Variables may be categorized further as qualitative or quantitative. Variables that yield observations on which individuals can be categorized according to some characteristic or quality are referred to as **qualitative variables.** Examples of qualitative variables are occupation, sex, marital status, and education level. Variables that yield observations that can be measured are considered to be **quantitative variables.** Examples of quantitative variables are weight, height, and serum cholesterol.

Table 3.1 Data for a Sample of 100 Individuals of the Honolulu Heart Study Population of 7683 Persons, 1969

ID	Educa-tional Level	Weight (KG)	Height (CM)	Age	Smoking Status	Physical Activity at Home	Blood Glucose	Serum Choles-terol	Systolic Blood Pressure	Ponderal Index
1	2	70	165	61	1	1	107	199	102	40.0361
2	1	60	162	52	0	2	145	267	138	41.3808
3	1	62	150	52	1	1	237	272	190	37.8990
4	2	66	165	51	1	1	91	166	122	40.8291
5	2	70	162	51	0	1	185	239	128	39.3082
6	4	59	165	53	0	2	106	189	112	42.3838
7	1	47	160	61	0	1	177	238	128	44.3358
8	3	66	170	48	1	1	120	223	116	42.0663
9	5	56	155	54	0	2	116	279	134	40.5138
10	2	62	167	48	0	1	105	190	104	42.1942
11	4	68	165	49	1	2	109	240	116	40.4248
12	1	65	166	48	0	1	186	209	152	41.2862
13	1	56	157	55	0	2	257	210	134	41.0365
14	2	80	161	49	0	1	218	171	132	37.3648
15	3	66	160	50	0	2	164	255	130	39.5918
16	4	91	170	52	0	2	158	232	118	37.7951
17	3	71	170	48	1	1	117	147	136	41.0547
18	5	66	152	59	0	2	130	268	108	37.6123
19	1	73	159	59	0	2	132	231	108	38.0444
20	4	59	161	52	0	1	138	199	128	41.3563
21	1	64	162	52	1	1	131	255	118	40.5001
22	3	55	161	52	1	1	88	199	134	42.3356
23	2	78	175	50	1	1	161	228	178	40.9582
24	2	59	160	54	0	1	145	240	134	41.0995
25	3	51	167	48	1	2	128	184	162	45.0326
26	3	83	171	55	0	1	231	192	162	39.2016
27	2	66	157	49	1	2	78	211	120	38.8495
28	4	61	165	51	0	1	113	201	98	41.9155
29	2	65	160	53	0	1	134	203	144	39.7939
30	3	75	172	49	0	1	104	243	118	40.7858
31	4	61	164	49	0	2	122	181	118	41.6615
32	1	73	157	53	1	2	442	382	138	37.5658
33	2	66	157	52	0	1	237	186	134	38.8495
34	1	73	155	48	0	2	148	198	108	37.0873
35	2	61	160	53	0	1	231	165	96	40.6453
36	3	68	162	50	0	2	161	219	142	39.6898
37	2	52	157	50	0	2	119	196	122	42.0629
38	5	73	162	50	0	1	185	239	146	38.7622
39	1	52	165	61	1	2	118	259	126	44.2062
40	1	56	162	53	1	1	98	162	176	42.3434
41	3	67	170	48	1	2	218	178	104	41.8560
42	1	61	160	47	0	1	147	246	112	40.6453
43	3	52	166	62	1	2	176	176	140	44.4741
44	2	61	172	56	1	2	106	157	102	43.6937
45	3	62	164	55	1	2	109	179	142	41.4362
46	2	56	155	57	1	2	138	231	146	40.5138
47	1	55	157	50	0	2	84	183	92	41.2838

Table 3.1 (*Continued*)

ID	Educa-tional Level	Weight (KG)	Height (CM)	Age	Smoking Status	Physical Activity at Home	Blood Glucose	Serum Choles-terol	Systolic Blood Pressure	Ponderal Index
48	3	66	165	48	1	2	137	213	112	40.8291
49	1	59	159	51	0	2	139	230	152	40.8426
50	3	53	152	53	1	2	97	134	116	40.4655
51	5	71	173	52	0	2	169	181	118	41.7792
52	2	57	152	49	0	1	160	234	128	39.4959
53	2	73	165	50	1	1	123	161	116	39.4800
54	3	75	170	49	0	2	130	289	134	40.3115
55	3	80	171	50	1	2	198	186	108	39.6856
56	4	49	157	53	0	1	215	298	134	42.9044
57	4	65	162	52	0	1	177	211	124	40.2913
58	2	82	170	56	0	2	100	189	124	39.1302
59	3	55	155	52	0	2	91	164	114	40.7578
60	3	61	165	58	0	1	141	219	154	41.9155
61	2	50	155	54	1	2	139	287	114	42.0735
62	5	58	160	56	0	1	176	179	114	41.3343
63	1	55	166	50	1	2	218	216	98	43.6503
64	5	59	161	47	0	2	146	224	128	41.3564
65	2	68	165	53	1	1	128	212	130	40.4248
66	2	60	170	53	1	2	127	230	122	43.4243
67	1	77	160	47	1	1	76	231	112	37.6089
68	5	60	155	52	0	1	126	185	106	39.5927
69	3	70	164	54	0	1	184	180	128	39.7935
70	2	70	165	46	0	1	58	205	128	40.0361
71	3	77	160	58	1	1	95	219	116	37.6089
72	5	86	160	53	0	2	144	286	154	36.2483
73	2	67	152	49	1	2	124	261	126	37.4242
74	3	77	165	53	1	1	167	221	140	38.7841
75	3	75	169	57	0	2	150	194	122	40.0744
76	2	70	165	52	0	2	156	248	154	40.0361
77	2	70	165	49	1	1	193	216	140	40.0361
78	1	71	157	53	0	1	194	195	120	37.9153
79	1	55	162	49	0	2	73	217	140	42.5985
80	2	59	165	53	1	2	98	186	114	42.3838
81	3	64	159	50	0	2	127	218	122	39.7501
82	1	66	160	54	0	1	153	173	94	39.5918
83	4	59	165	60	0	2	161	221	122	42.3838
84	3	68	165	57	0	1	194	206	172	40.4248
85	5	58	160	52	0	1	87	215	100	41.3343
86	1	57	154	65	1	1	188	176	150	40.0156
87	2	60	160	65	0	2	149	240	154	40.5699
88	2	53	162	62	0	1	215	234	170	43.1277
89	2	61	159	62	1	2	163	190	140	40.3913
90	1	66	154	62	0	1	111	204	144	38.1072
91	1	61	152	67	0	2	198	256	156	38.6131
92	2	52	152	66	0	2	265	296	132	40.7233
93	1	59	155	62	0	2	143	223	140	39.8151
94	1	63	155	62	1	1	136	225	150	38.9540
95	2	61	165	63	0	2	298	217	130	41.9155

Table 3.1 (*Continued*)

ID	Educational Level	Weight (KG)	Height (CM)	Age	Smoking Status	Physical Activity at Home	Blood Glucose	Serum Cholesterol	Systolic Blood Pressure	Ponderal Index
96	2	68	155	67	0	2	173	251	118	37.9749
97	1	58	170	62	0	1	148	187	162	43.9178
98	3	68	160	55	0	1	110	290	128	39.1998
99	5	60	159	50	0	2	188	238	130	40.6144
100	2	61	160	54	1	1	208	218	208	40.6453

Code for variables:
Education: 1 = none, 2 = primary, 3 = intermediate, 4 = senior high, 5 = technical school,
  6 = university
Weight: in kilograms
Height: in centimeters
Smoking: 0 = no, 1 = yes
Physical activity: 1 = mostly sitting, 2 = moderate, 3 = heavy
Blood glucose: in milligrams percent
Serum cholesterol: in milligrams percent
Systolic blood pressure: in millimeters of mercury
Ponderal index: height $\div \sqrt[3]{\text{weight}}$

Quantitative variables can be classified further as **discrete** or **continuous.** The number of children per household, the number of times you visit a doctor, and the number of missing teeth are termed discrete variables; they must always be integers—that is, whole numbers (e.g., 0, 1, and 2). Variables such as age, height, and weight may take on fractional values (e.g., 37.8, 138.2, and 112.9). They are referred to as continuous variables.

Statisticians often treat discrete variables as continuous variables. An example that you probably have noticed is the number of children per household. You may see a number such as 2.4 children per household. Obviously you cannot have .4 of a child, yet this is a widely used statistic. The reason for treating discrete variables as continuous variables is that it significantly improves the accuracy or predictability of the data. If a community group, such as a school, is trying to estimate the number of children that will need services, how should that estimate be made? Let us assume that a community anticipated that it would have 100 new households in the next 5 years. If the number of children per household is treated strictly as a discrete variable, then the average number of children per household would be 2 and the estimate for 100 new households would be an increase of 200 children. Treating the discrete variable as a continuous variable (2.4 children per household) would yield an estimate of 240 children. In all likelihood, the 240 would be the more accurate estimate.

Different types of variables are analyzed differently. Know what type of data you have. This will help you to select quickly the appropriate method of analyis.

.3   THE FREQUENCY TABLE

Perhaps the most convenient way of summarizing or displaying data is by means of a **frequency table.** Tables 3.2 and 3.3 are examples of frequency tables, constructed from the systolic blood pressure readings (by smoking status) of our Honolulu Heart Study sample, Table 3.1. The first step in constructing such a table is to compute the interval spanned by the data. We can obtain this interval by arranging the data into an **array,** a listing of all observations from smallest to largest. We find that the overall blood pressure interval is 92–208 mm, a range of 116 mm.

The next step is to divide the range into a number of arbitrary but usually equal and nonoverlapping segments called **class intervals.** Intervals are usually

Table 3.2 Frequency Table for Systolic Blood Pressure of *Nonsmokers* from Table 3.1

Class Interval (Systolic Blood Pressure*)	Tally	f (Frequency)	Relative Frequency (%)
90–109	ⅢⅠ ⅢⅠ	10	16
110–129	ⅢⅠ ⅢⅠ ⅢⅠ ⅢⅠ ⅢⅠ	24	38
130–149	ⅢⅠ ⅢⅠ ⅢⅠ ⅢⅠ	18	29
150–169	ⅢⅠ ⅢⅠ	9	14
170–189	Ⅱ	2	3
190–209		0	0
Total		63	100

SOURCE: Honolulu Heart Study.

*In millimeters of mercury.

Table 3.3 Frequency Table for Systolic Blood Pressure of *Smokers* from Table 3.1

Class Interval (Systolic Blood Pressure*)	Tally	f (Frequency)	Relative Frequency (%)
90–109	ⅢⅠ	5	14
110–129	ⅢⅠ ⅢⅠ ⅢⅠ	15	41
130–149	ⅢⅠ ⅢⅠ	10	27
150–169	ⅢⅠ	3	8
170–189	Ⅱ	2	5
190–209	Ⅱ	2	5
Total		63	100

SOURCE: Honolulu Heart Study.

*In millimeters of mercury.

equal in length, thereby aiding the comparisons between the frequencies of any two intervals. The beginning and length of the class intervals should be reasonably convenient and correspond, as far as possible, to meaningful stopping points. The number of intervals depends, of course, on the number of observations but in general should range from 5 to 15. With too many class intervals, the data are not summarized enough for a clear visualization of how they are distributed. With too few, the data are oversummarized and some of the details of the distribution may be lost. Suppose we decide that we want six intervals. Remember, the size of intervals in general should range from 5 to 15. In this case, however, the size of the class interval should be $116/6 = 19.3$ or 20. In Table 3.2, therefore, the first interval is 90–109 mm, where 90 and 109 are called **class limits.** The number of observations falling into any given interval is called the **class frequency,** usually symbolized by f. For the first interval, f is 10, obtained from the tally. The **tally** is a familiar and convenient way of keeping score of a set of observations.

A completed frequency table provides a frequency distribution. A **frequency distribution** is a table (or a graph or an equation) that includes a set of intervals and displays the number of measurements in each interval—that is, it shows the proportion of a population or sample having certain characteristics. From Tables 3.2 and 3.3 we can derive the range, the frequency in each of the intervals, and the total number of observations collected. Frequency tables should include an appropriate descriptive title, specify the *units of measurement,* and cite the source of data.

Frequency tables often include other features, for example, the **relative frequency,** which represents the relative percentage to total cases of any class interval. It is obtained by dividing the number of cases in the class interval by the total number of cases and multiplying by 100. For example, in Table 3.2, the relative frequency of the first class, 90–109 mm, is $(10/63)100 = 16\%$. It indicates the percentage of total cases that fall in a given class interval. The use of relative frequency is particularly helpful in making a comparison between two sets of data that have a different number of observations, like our 63 nonsmokers and 37 smokers. For example, in the blood pressure range of 90–109 mm, 10 (16%) of the nonsmokers and 5 (14%) of the smokers were represented.

Class boundaries are points that demarcate the true upper limit of one class and the true lower limit of the next. For example, the class boundary between classes 90–109 and 110–129 is 109.5; it is the upper boundary for the former and the lower boundary for the latter. Class boundaries may be used in place of class limits.

Cumulative relative frequency, also known as **cumulative percentage,** gives that percentage of individuals having a measurement less than or equal to the upper boundary of the class interval. The cumulative percentage distribution is of value in obtaining such commonly used statistics as the median and percentile scores, which we will discuss later in this chapter. It also makes possible a rapid comparison of entire frequency distributions, ruling out any

Table 3.4 Comparison of Systolic Blood Pressure Between Smokers and Nonsmokers from Table 3.1

Class Interval (Systolic Blood Pressure*)	Relative Frequency (%)		Cumulative Relative Frequency (%)	
	Nonsmokers	Smokers	Nonsmokers	Smokers
90–109	16	14	16	14
110–129	38	41	54	55
130–149	29	27	83	82
150–169	14	8	97	90
170–189	3	5	100	95
190–209	0	5	100	100

SOURCE: Honolulu Heart Study.

*In millimeters of mercury.

need to compare individual class intervals. Cumulative relative frequency is easy to compute. You do it by successively cumulating the relative frequencies of each of the various class intervals. In our example, for nonsmokers the cumulative percentage for the first four intervals is 16 + 38 + 29 + 14 = 97% (Table 3.4). The interpretation: 97% of the nonsmokers in the sample have a systolic blood pressure below 169.5. By comparison, 90% of the smokers have a blood pressure below the same level. An alternate way of looking at this is to note that 3% of the nonsmokers and 10% of the smokers have a systolic blood pressure above 169.5.

4 GRAPHING DATA

The second way of displaying data is by use of **graphs.** Graphs give the user a nice overview of the essential features of the data. Although such visual aids are even easier to read than tables, they often do not give the same detail.

Graphs are designed to help the user obtain at a glance an intuitive feeling for the data. So it is essential that each graph be self-explanatory—that is, have a descriptive title, labeled axes, and an indication of the units of observation. An effective graph is simple and clean. It should not attempt to present so much information that it is difficult to comprehend. Seven graphs will be discussed here—namely, histograms, frequency polygons, cumulative frequency polygons, stem-and-leaf displays, bar charts, pie charts, and box and whisker plots.

Histograms

Perhaps the most common graph is the histogram. A histogram is nothing more than a pictorial representation of the frequency table. It consists of an **abscissa** (horizontal axis), which depicts the class boundaries (not limits), and a perpen-
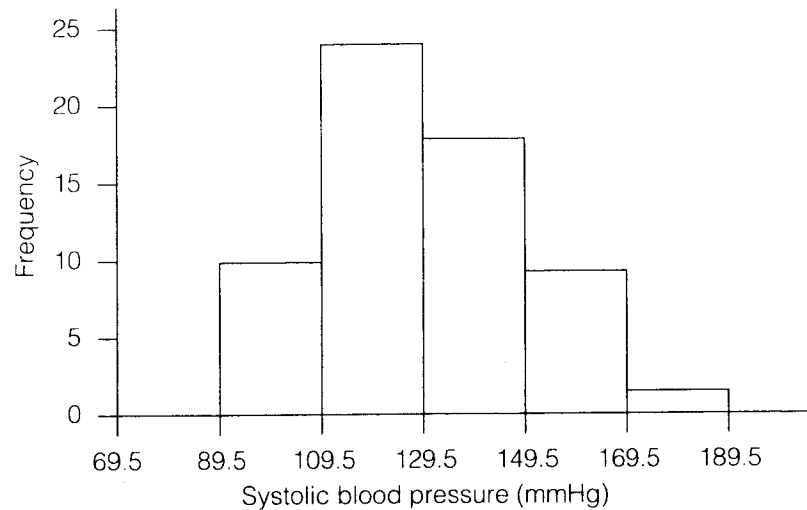
Figure 3.1 Histogram Illustrating the Data of Table 3.2: Systolic Blood Pressure of a Sample of 63 Nonsmokers from the Honolulu Heart Study

dicular **ordinate** (vertical axis), which depicts the frequency (or relative frequency) of observations. <u>The vertical scale should begin at zero.</u> A general rule in laying out the two scales is to make the height of the vertical scale equal to approximately three-fourths the length of the horizontal scale. Otherwise, the histogram may appear to be out of proportion with reality. Once the scales have been laid out, a vertical bar is constructed above each class interval equal in height to its class frequency. For our Honolulu Heart Study example, the bar over the first class interval is 10 units high (Figure 3.1).

Frequencies are represented not only by height but also by the area of each bar. The total area represents 100%. From Figure 3.1 it is possible to measure that 16% of the area corresponds to the 10 scores in the class interval 89.5–109.5 and that 38% of the area corresponds to the 24 observations in the second bar. Because area is proportional to the number of observations, be especially careful when constructing histograms from frequency tables that have unequal class intervals. How this is done is illustrated with the income data shown in Table 3.5.

From Table 3.5 we can see that the first five class intervals are measured in $5000 units while the next two intervals are $10,000 (i.e., two $5000 units) and $15,000 (three $5000 units), respectively. Because area is an indication of frequency in a histogram, we have to allocate the appropriate amount of area to each bar. The heights of the first five class intervals are their respective relative frequencies—that is, 6.9, 11.5, and so on. The height for the other intervals is obtained using the following formula:

Height = relative frequency/interval width

Table 3.5 Household Income, 1989

Income ($)	Number of Households	Relative Frequency (%)
0–4,999	6,320,400	6.9
5,000–9,999	10,534,000	11.5
10,000–14,999	9,709,600	10.6
15,000–19,999	9,100,000	10.0
20,000–24,999	8,427,200	9.2
25,000–34,999	14,747,600	16.1
35,000–49,999	15,755,200	17.2
50,000–74,999	16,488,000	18.0
75,000 and over	458,000	0.5
Total	91,600,000	100.0

The height for the sixth interval is 8.05 (= 16.1/2) and for the seventh, 5.7 (= 17.2/3).

For the 50,000 to 75,000 interval, determining the width of the interval becomes tricky. In this case, our interval will be five times wider than the $5000 interval [(75,000 − 50,000)/5000 = 5]. Consequently, the height for the last interval will be 3.6 (= 18.0/5).

Using these heights, we can now draw the histogram, as shown in Figure 3.2. From Figure 3.2 we can see that the percent frequencies of households decreases as income increases. Furthermore, we can say that there is a higher percentage of households with low income than with high income.

Frequency Polygons

A second commonly used graph is the **frequency polygon**, which uses the same axes as the histogram. It is constructed by marking a point (at the same height
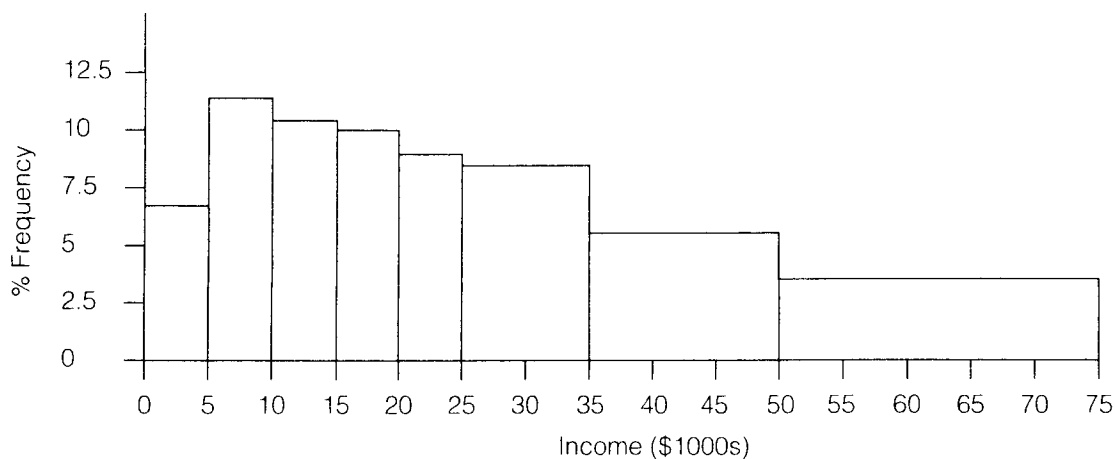
Figure 3.2 Histogram of Household Income, 1989, United States
*The 0.5% of households with income $75,000 and over is not shown.
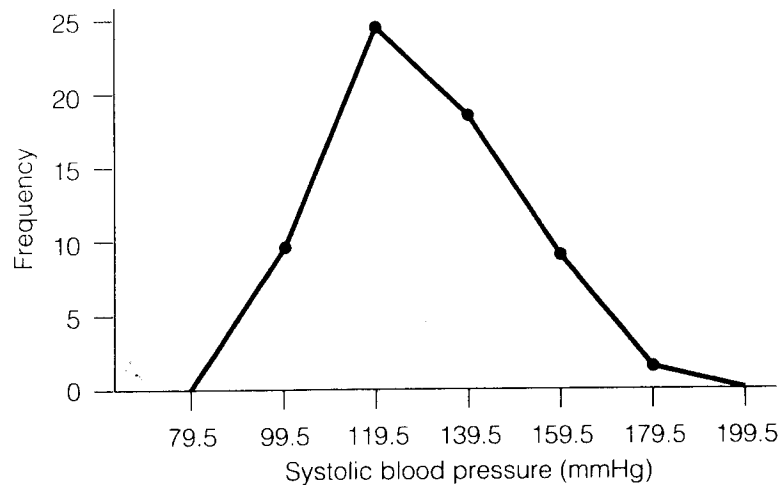
Figure 3.3   Frequency Polygon Illustrating the Data of Table
3.2: Systolic Blood Pressure of a Sample of 63 Nonsmokers
from the Honolulu Heart Study

as the histogram's bar) at the **midpoint** of the class interval. These points are
then connected with straight lines. At the ends, the points are connected to the
midpoints of the previous (and succeeding) intervals of zero frequency (Figure
3.3). Frequency polygons, especially when superimposed, are superior to his-
tograms in providing a means of comparing two frequency distributions. In fre-
quency polygons, the frequency of observations in a given class interval is rep-
resented by the area contained beneath the line segment and within the class
interval. This area is proportional to the total number of observations in the fre-
quency distribution. Frequency polygons should be used to graph only quanti-
tative (numerical) data, never qualitative (i.e., nominal or ordinal) data because
these latter data are not continuous.

Frequency polygons may take on a number of different shapes. Some of
those most commonly encountered are shown in Figure 3.4. Part (a) of the fig-
ure is the classic "bell-shaped" **symmetrical distribution.** Part (b) is a **bimodal**
(having two peaks) **distribution** that could represent an overlapping group of
males and females. Part (c) is a **rectangular distribution** in which each class in-
terval is equally represented. Parts (a) and (c) are symmetrical, whereas parts
(d) and (e) are **skewed,** or asymmetrical. The frequency polygon of part (d) is
positively skewed since it tapers off in the positive direction, and part (e) is neg-
atively skewed.

Cumulative Frequency Polygons

At times it is useful to construct a **cumulative frequency polygon,** also called
an **ogive,** which is a third type of graph. Although the horizontal scale is the
same as that used for a histogram, the vertical scale indicates cumulative

| (a) Symmetrical | (b) Bimodal | (c) Rectangular |

| (d) Skewed to right | (e) Skewed to left |
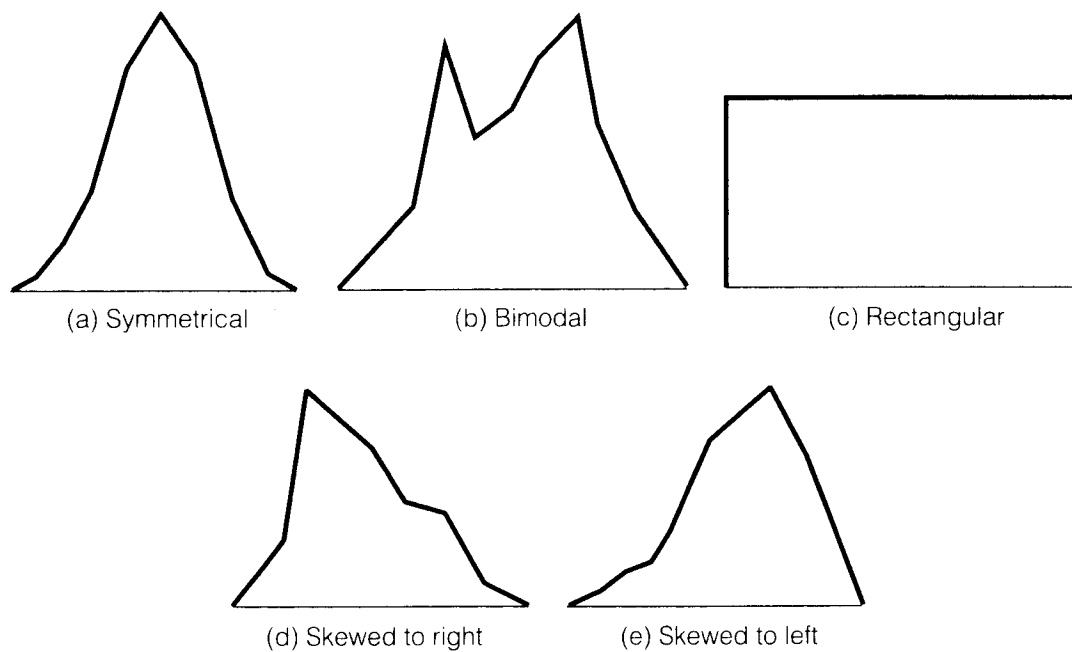
Figure 3.4   Various Shapes of Frequency Polygons

frequency or cumulative relative frequency. To construct the ogive, we place a point at the upper class boundary of each class interval. Each point represents the cumulative relative frequency for that class. Note that not until the upper class boundary has been reached have all the data of a class interval been accumulated. The ogive is completed by connecting the points (Figure 3.5). Ogives are useful in comparing two sets of data, as, for example, data on healthy and diseased individuals. In Figure 3.5 we can see that 90% of the nonsmokers and 86% of the smokers had systolic blood pressures below 160 mmHg. The ogive gives for each interval the cumulative relative frequency—that is, the percentage of cases having systolic blood pressures in that interval or a lower one.

Percentiles may be obtained from an ogive. The 90th percentile is that observation that exceeds 90% of the set of observations and is exceeded by only 10% of them. Percentiles are readily obtained, as in Figure 3.5. In our example, the 50th percentile, or median, for nonsmokers, is a blood pressure of 127.5 mmHg, and the 90th percentile is 159.5 mmHg.

Stem-and-Leaf Displays

Tukey (1977) has suggested an innovative technique for summarizing data that utilizes characteristics of the frequency distribution and the histogram. It is referred to as the **stem-and-leaf display;** in this technique, the "stems" represent the class intervals and the "leaves" are the strings of values within each class interval. Table 3.6 illustrates the usefulness of this technique in helping you develop a better feel for your data. The table is a stem-and-leaf display that utilizes the observations of systolic blood pressures of the 63 nonsmokers of
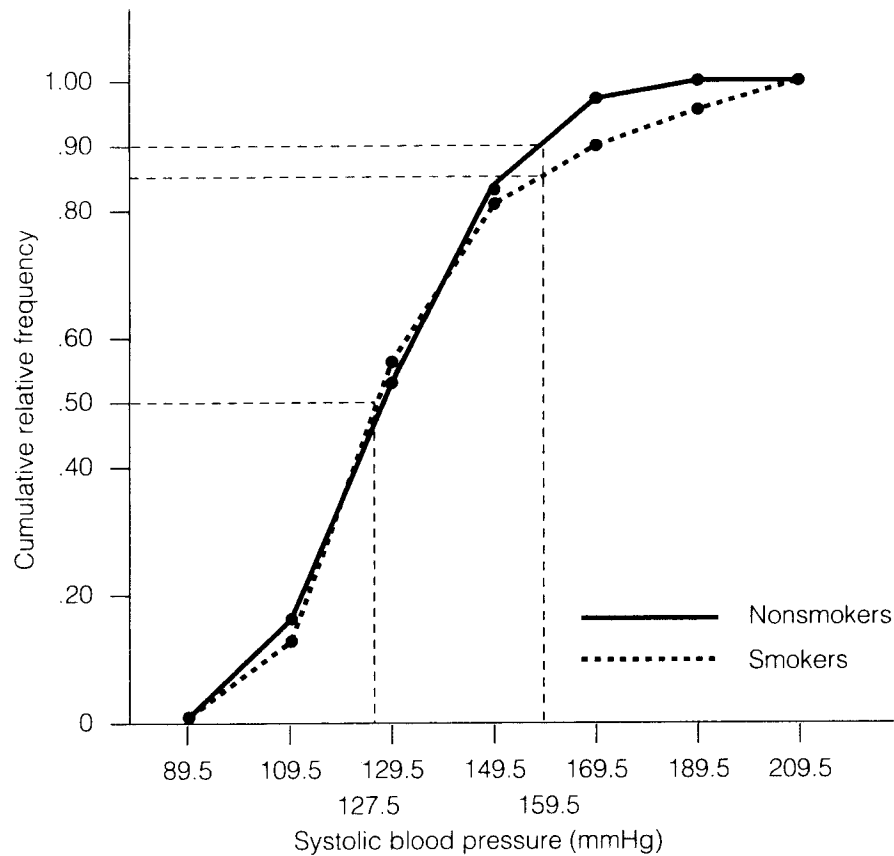
Figure 3.5 Ogives Illustrating the Data of Table 3.4: Systolic Blood Pressure of a Sample of 63 Nonsmokers and 37 Smokers from the Honolulu Heart Study

Table 3.2. For each stem (interval) we arrange the last digits of the observations from the lowest to the highest. This arrangement is referred to as the leaf. The leaves (strings of observations) portray a histogram laid on its side; each leaf reflects the values of the observations, from which it is easy to note their size and frequencies. Consequently, we have displayed all observations and provided a visual description of the shape of the distribution. It is often useful to present the stem-and-leaf display together with a conventional frequency distribution. From the stem-and-leaf display of the systolic blood pressure data (Table 3.6) we can see that the range of measurements is 92 to 172. The measurements in the 120s occur most frequently, with 128 being the most frequent. We can also see which measurements are not represented.

Bar Charts

The **bar chart** is a convenient graphical device that is particularly useful for displaying nominal or ordinal data—data like ethnicity, sex, and treatment category. The various categories are represented along the horizontal axis. They may be arranged alphabetically, by frequency within a category, or on some

Table 3.6 Stem-and-Leaf Display of Systolic Blood Pressure of 63 *Nonsmokers* (Data from Table 3.2)

Stems (Intervals)	Leaves (Observations)	Frequency (f)
90–99	2 4 6 8	4
100–109	0 4 6 8 8 8	6
110–119	2 2 4 4 8 8 8 8 8	9
120–129	0 2 2 2 2 4 4 8 8 8 8 8 8 8 8	15
130–139	0 0 0 2 2 4 4 4 4 4 4 8	12
140–149	0 0 2 4 4 6	6
150–159	2 2 4 4 4 4 6	7
160–169	2 2	2
170–179	0 2	2
180–189		0
Total		63

other rational basis. We often arrange bar charts according to frequency, beginning with the most frequent and ending with the least frequent. The height of each bar is equal to the frequency of items for that category. To prevent any impression of continuity, it is important that all the bars be of equal width and separate, as in Figure 3.6.

Note that in a *bar chart*, relative frequencies are shown by *heights*, but in a *histogram*, relative frequencies are shown by the *areas* within the bars.

To avoid misleading a reader it is essential that the scale on the vertical axis begin at zero. If that is impractical, one should employ broken bars (or a similar device), as shown in Figure 3.7. Here is an example of what can happen if
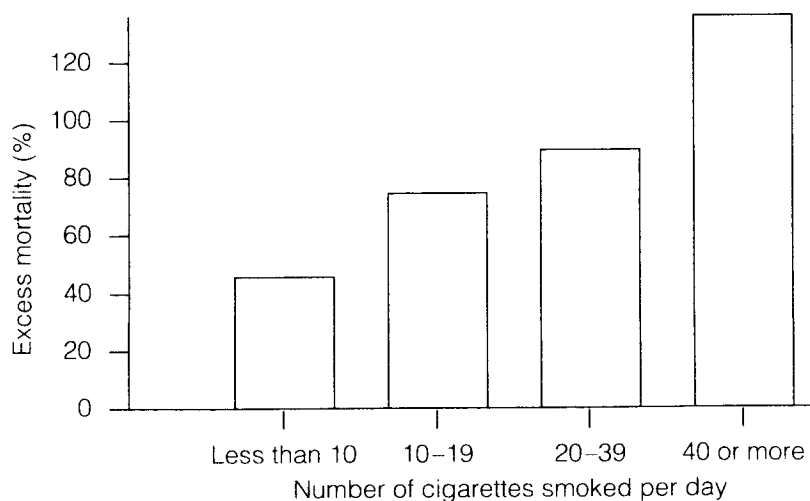
Figure 3.6 Bar Chart of Excess Mortality of Smokers over Nonsmokers According to Number of Cigarettes Smoked. SOURCE: Hammond (1966).
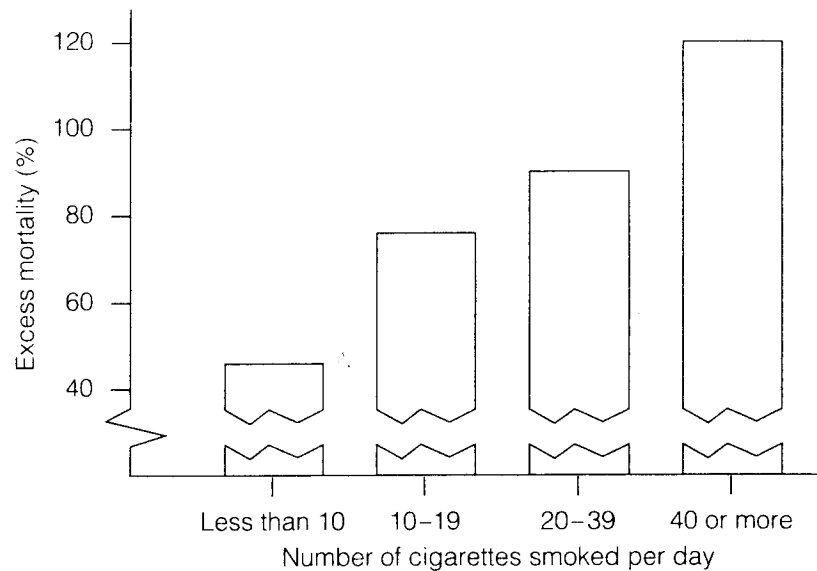
Figure 3.7   Bars Broken to Show Vertical Scale Does Not Begin at
Zero.   SOURCE: Hammond (1966).

neither procedure is followed. The public relations department of a West Coast
college recently circulated the graph shown in Figure 3.8a. It gives the clear im-
pression that enrollment doubled between 1986 and 1992. The reason for this is
that the bars begin not at zero but at 2000. Persons unskilled in interpreting
graphical data may find themselves drawn into one of the many pitfalls that
are so well documented in books on the misuse of statistics. Figure 3.8b illus-
trates the correct way of presenting the same enrollment statistics. This graph
makes clear that the enrollment increased by only about 50% over the seven
years.

Pie Charts

A common device for displaying data arranged in categories is the **pie chart**
(Figure 3.9), a circle divided into wedges that correspond to the percentage fre-
quencies of the distribution. Pie charts are useful in conveying data that consist
of a small number of categories.

Box and Whisker Plots

At times we may wish to graphically examine data such as long distance tele-
phone charges for different cities to get an idea about the typical customer and
the range of the billings. We can do this by using a **box and whisker plot.** To do
so we need to determine the median and the quartile statistics.

The **median** is the score that divides a ranked series of scores into two equal
halves. If there is an equal number of scores you will need to obtain the average
(mean) of the two middle scores.
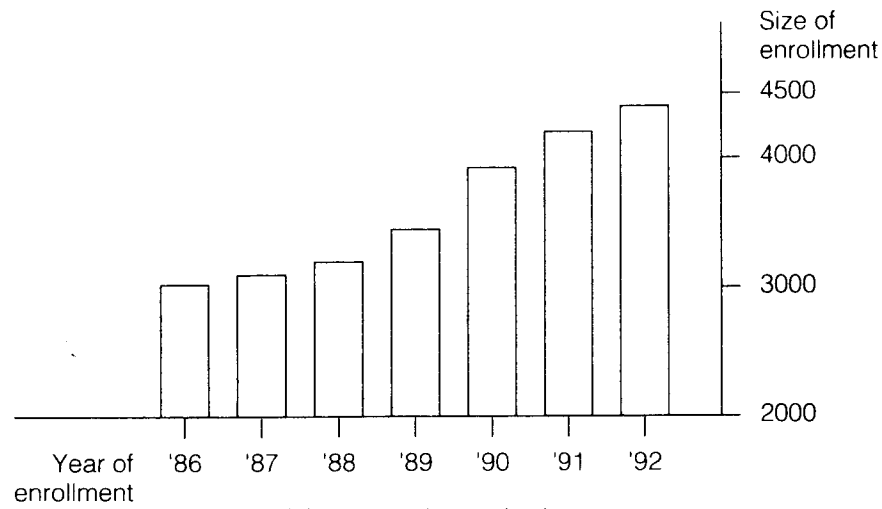
(a) Incorrectly graphed
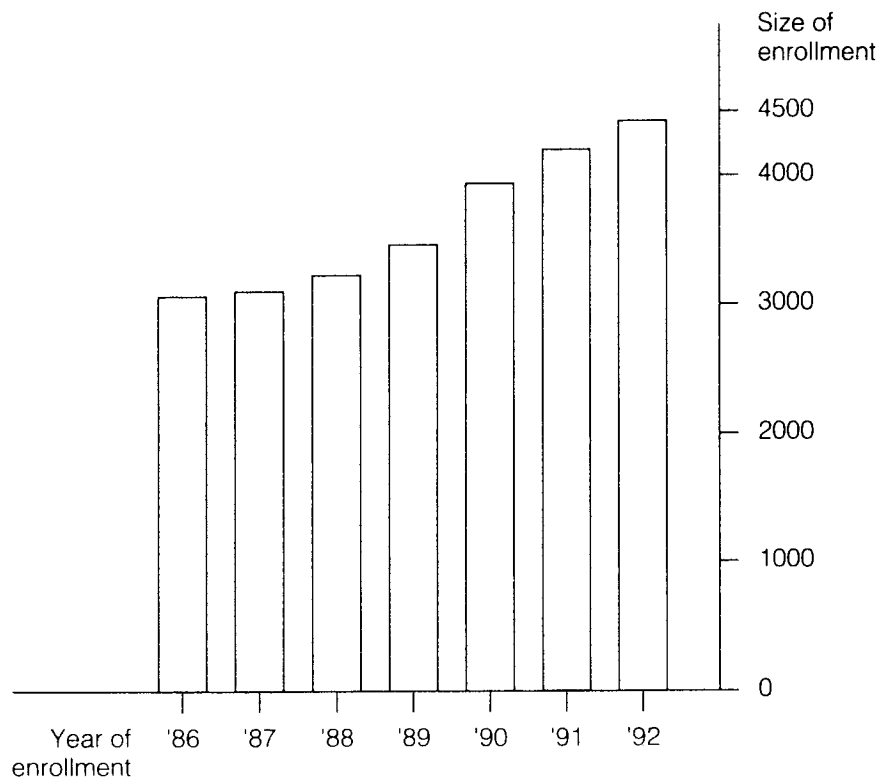
(b) Correctly graphed

Figure 3.8   Size of Enrollment of a West Coast College, 1986 to 1992

Here is an example. Determine the sample median for the two samples:

A: 26, 27, 31, 32, 35, 38, 39, 40, 41   Median = 35

B: 15, 16, 17, 18, 19, 21, 22, 25, 29, 30   Median = 19 + 21 divided by
                                                              2 = 20
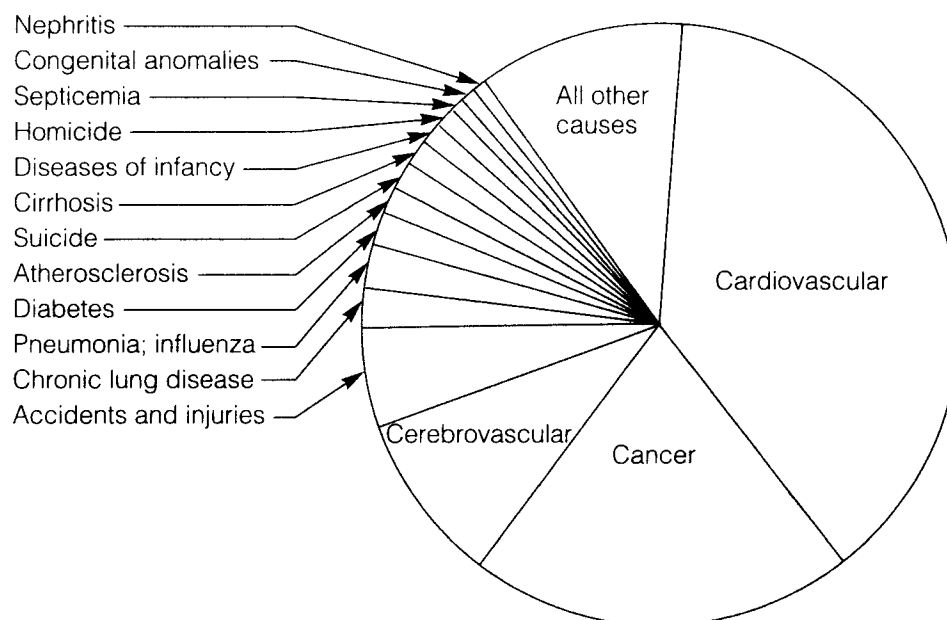
Figure 3.9   Pie Chart of Leading Causes of Death in the United States, 1987.
SOURCE: National Center for Health Statistics (1990).

Half of the scores in each sample are less than the median and half are larger than the median. To determine the quartiles we need to divide the scores into four equal groups.

In Figure 3.10 (page 40) we see that we use only five values to summarize the data: the two extremes and the three quartiles. Even with such a considerable condensation, the plot provides interesting information about the sample. The two ends of the box show the range within which the middle 50% of all the measurements lie. The median is the center dot of the sample data and the ends of the whiskers show the spread of the data.

Conclusion

The principles of tabulating and graphing data are essential if we are to understand and evaluate the flood of data with which we are bombarded. By proper use of these principles, the statistician is able to present data accurately and lucidly. It is also important to know which method of presentation to choose for each specific type of data. Tables are usually comprehensive but do not convey the information as quickly or as impressively as do graphs. Remember that graphs and tables must tell their own story and stand on their own. They should be complete in themselves and require little (if any) explanation in the text.
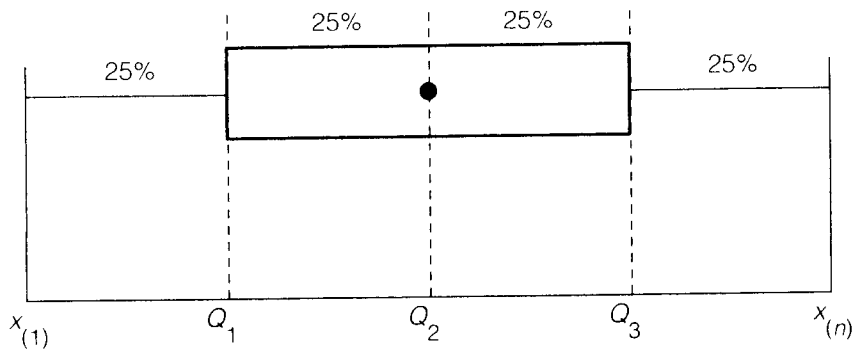
	For sample A the data are:	For sample B the data are:
x_1 =	26	15
Q_1 =	31.5	17
Median =	35	20
Q_2 =	39.5	25
x_n =	41	30

Figure 3.10 Summary of Telephone Charge Data Using a Box and Whisker Plot

Vocabulary List

abscissa
array
bar chart
bimodal distribution
box and whisker plot
class boundaries
class frequency
class interval
class limits
class midpoint
continuous variable
cumulative frequency
   polygon (ogive)
cumulative percentage

cumulative relative
   frequency
discrete variable
frequency distribution
frequency polygon
frequency table
graph
histogram
interval scale
median
midpoint (class
   midpoint)
nominal number
ordinal number

ordinate
percentile
pie chart
qualitative variable
quantitative variable
ratio scale
rectangular distribution
relative frequency
skewed distribution
stem-and-leaf display
symmetrical
   distribution
tally

Exercises

3.1   Refer to the variables of Table 3.1.
a. Classify each variable as to whether it is qualitative or quantitative.
b. Which of the quantitative variables are discrete? Which are continuous?
c. Name an appropriate type of graph for presenting each variable.

d. Name a discrete variable that one might be interested in measuring for the Honolulu Heart Study group.

3.2 Name the variables represented in Table 2.2 and state which type each is.

3.3 How would you describe the shape of Figure 3.3? Refer to Table 3.4 and state whether the distribution of smokers' systolic blood pressures would be similar in shape to that of nonsmokers.

3.4 State the principal difference between a negatively skewed distribution and a positively skewed one.

3.5 a. From the 83 observations of diastolic blood pressure in Table 2.2, prepare a frequency table like Table 3.2 that includes class interval, class frequency, relative frequency, cumulative relative frequency, and class midpoint.
b. Using the same sheet of graph paper, draw a histogram and a frequency polygon for the same data.
c. Construct an ogive for the same data.
d. Find the following percentiles from the ogive: 20th, 50th (median), and 90th.
e. What percentage of the observations are less than 70? 80? 90?

3.6 For the serum cholesterol values of Table 3.1, perform the same operations as suggested in (a) and (b) of Exercise 3.5. Do this by activity status; that is, for those who reported their physical activity as (a) mostly sitting (code 1) and (b) moderate (code 2), make separate frequency tables, histograms, and frequency polygons for the serum cholesterol values.

3.7 Make a bar graph of the educational levels of Table 3.1.

3.8 With each of the variables listed here, two graphical methods are mentioned. Indicate which method is more appropriate. State why one method is more appropriate than the other.
a. number of dental cavities per person: pie chart, bar graph
b. triglyceride level: frequency polygon, bar graph
c. occupational classification: pie chart, histogram
d. birthrate by year: line graph, histogram

3.9 Prepare a stem-and-leaf display for the weights listed in Table 3.1.
a. Which are the smallest and the largest weights?
b. Which is the most frequent weight?

3.10 a. Prepare a stem-and-leaf display for the systolic blood pressure measurements of smokers in Table 3.1. Use the same stems as in Table 3.6, but put the leaves on the left side of the stem.
b. Combine the stem-and-leaf displays of Exercise 3.10a and Table 3.6 into a back-to-back stem display and compare the two distributions.

3.11 Prepare a stem-and-leaf display for the heights listed in Table 3.1.
a. Which is the smallest and which is the largest height?
b. Which is the most frequent height?

3.12 For the weight data in Table 3.1, do the following:
a. Construct separate frequency tables for smokers and for nonsmokers. Use six equal class intervals beginning with 45.
b. Construct a histogram for each group.

   c. Construct a frequency polygon for each group on the same graph.

   d. Compare and discuss the differences in the frequency distributions between smoker and nonsmoker weights.

   e. Construct an ogive for each group. Estimate its 50th percentile and compare them for the two groups.

3.13 Construct a bar chart of educational level using the data in Table 3.1 for

   a. smokers

   b. nonsmokers

   c. Compare the two bar charts and comment.

3.14 For the serum cholesterol data in Table 3.1, use equal class intervals of 30 beginning with 130, to construct

   a. a separate frequency table for each of the three subgroups classified on physical activity

   b. a histogram for each subgroup

   c. a frequency polygon for each of the three groups. Compare and discuss the differences in the three frequency polygons.

   d. an ogive for each group. Estimate its 50th percentile and compare the three.

3.15 Prepare a pie chart of the educational level for the entire sample listed in Table 3.1.

3.16 a. Using the income data from Table 3.5, combine the first two and also the third and fourth class intervals and prepare a histogram similar to Figure 3.2.

   b. Compare your histogram with that of Figure 3.2 and describe your findings.

3.17 The following are weight losses (in pounds) of 25 individuals who enrolled in a five-week weight-control program:

```
9, 7, 10, 11, 10, 2, 3, 11, 5
4, 8, 10, 9, 12, 5, 4, 11, 8
3, 6, 9, 7, 4, 8, 9
```

   a. Construct a frequency table with these six class intervals: 2–3, 4–5, 6–7, 8–9, 10–11, 12–13 each.

   b. Construct a histogram of the weight losses.

   c. Construct a frequency polygon and describe the shape of the frequency distribution.

   d. What might be a possible interpretation of the particular shape of this distribution?

   e. What was the most common weight loss?

3.18 Compare the three frequency distributions that you constructed in Exercise 3.14 and describe them with regard to symmetry, skewness, and modality (most frequently occurring observation).

3.19 Classify the following data as either (1) nominal, (2) ordinal, (3) interval, or (4) ratio.

   a. names of students in this class

   b. the number of students in this class

   c. your 10 favorite songs

d. height

e. heads and tails on a coin

3.20 Briefly explain why discrete (discontinuous) variables are treated as continuous variables. Use an example as part of your explanation.

3.21 Given is the following grouped frequency distribution:

90–99

80–89

70–79

Answer the following questions:

a. What is the class interval?

b. What are the true limits for the interval 80–89?

Determine the median and quartiles necessary to construct a box and whisker plot for the following sets of data in Exercises 3.22 and 3.23.

3.22 3, 4, 7, 5, 4, 6, 4, 5, 8, 3, 4, 5, 6, 5, 4

3.23 18, 14, 17, 22, 16, 26, 33, 27, 35, 28, 44, 40, 31, 53, 70, 73, 62, 74, 93, 103, 75, 86, 84, 90, 79, 99, 73

3.24 Construct the box and whisker plots for the data in (a) Exercise 3.22 and (b) Exercise 3.23.

3.25 Using the following data found recently in FBI Uniform Crime Reports, construct a pie chart indicating the weapons used in committing these murders.

11,381 committed with firearms

3,957 committed with personal weapons such as hands or feet

1,099 committed with knives

19,257 all murders

3.26 Construct a box plot for the sample of $n = 100$ blood pressure readings listed in Table 3.1 separately for smokers and nonsmokers and provide a written comparison of the two groups based on the box plots.

3.27 Determine a box and whisker plot for weight loss data shown in Exercise 3.17.

4 Summarizing Data

Chapter Outline

4.1 Measures of Central Tendency
Describes, illustrates, and contrasts three common measures of central tendency—mean, median, and mode

4.2 Measures of Variation
Describes several measures of variation or variability, including the standard deviation

4.3 Coefficient of Variation
Defines the coefficient of variation, useful in comparing levels of variation

4.4 Means and Standard Deviations of a Population
Contrasts the equations for the parameters of a population with the statistics of a sample

Learning Objectives

After studying this chapter, you should be able to

1. Compute and distinguish between the uses of measures of central tendency: mean, median, and mode

2. Compute and list some uses for measures of variation: range, variance, and standard deviation

3. Compare sets of data by computing and comparing their coefficients of variation

4. Select the correct equations for computing the mean and the standard deviation

5. Be able to compute the mean and the standard deviation for grouped and un-grouped data

6. Understand the distinction between the population mean and the sample mean

MEASURES OF CENTRAL TENDENCY

Suppose you are considering accepting a new job with a well-known company. Salary is foremost in your mind, so you ask, "What is an employee's typical annual salary?" One person tells you, "$38,000"; another, "$30,000." You decide to check further into these inconsistent responses. Finally, you obtain some information you regard as reliable. Specifically, you are interested in knowing the lowest and the highest salaries, the typical salary, and relative frequencies of the various annual salaries. A small but representative sample of salaries shows them to be $26,000, $30,000, $30,000, $34,000, and $70,000. With this information at hand, you are now prepared to describe the salaries in the company. But to do this, you need to know how to compute statistics that characterize the center of the frequency distribution.

Given a set of data, one invariably wishes to find a value about which the observations tend to cluster. The three most common values are the mean, the median, and the mode. They are known as measures of **central tendency**—the tendency of a set of data to center around certain numerical values.

The Mean

The arithmetic mean (or, simply, **mean**) is computed by summing all the observations in the sample and dividing the sum by the number of observations. As there are other means, such as the harmonic and geometric means, it is essential to designate which type of mean one uses. In this text we use only the arithmetic mean.

Symbolically, the mean is represented by

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} \tag{4.1}$$

or

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{4.2}$$

In these expressions the symbol $\bar{x}$, representing the sample mean, is read "x-bar"; x_1 is the first and x_i the ith in a series of observations. In this text we use X (uppercase) to denote a random variable, and $\bar{x}$ (lowercase) to indicate a particular value of a function. The symbol Σ is the uppercase Greek letter sigma and denotes "the sum of." Thus

$$\sum_{i=1}^{n}$$

indicates that the sum is to begin with $i = 1$ and increment by one up to and including the last observation n.

For the sample of the five salaries,

$$x = \frac{\$26,000 + \$30,000 + \$30,000 + \$34,000 + \$70,000}{5} = \$38,000$$

The arithmetic mean may be considered the balance point, or fulcrum, in a distribution of observations. It considers the magnitude of each observation and is the point that balances the positive and negative deviations from the fulcrum. The mean is affected by the value of each observation of the distribution. Therefore, large values influence the mean and may distort it so that it no longer is representative of the typical values of a distribution.

The Median

In a list ranked according to size—that is, the observations arranged in an array—the **median** is the observation that divides the distribution into equal parts. The median is considered the most typical observation in a distribution. It is that value above which there are the same number of observations as below. In short, it is the middlemost value. In our example of five salaries, the median is $30,000. For an even number of observations, the median is the average of the two middlemost values.

The Mode

The **mode** is the observation that occurs most frequently. In the salary example, the mode is equal to $30,000. It can be read from a graph as that value on the abscissa that corresponds to the peak of the distribution. Frequency distributions, like the one displayed in Figure 3.5(b), are bimodal; that is, they have two modes. If all the values are different, there is no mode.

Which Average Should You Use?

With a bit of experience, you can readily determine which measure of central tendency is appropriate to a given situation. The arithmetic mean is by far the most commonly used. Because it considers, for example, the average amount of product consumed by a user, it is indispensable in business and commerce. If, for example, the average per capita consumption of sugar per year is 25 lb, then the amount of sugar to be sold in a town of 10,000 people would be 250,000 lb. Knowing the mean of a distribution also permits one to compare different frequency distributions.

If you want to know a typical observation in a distribution, particularly if it is skewed, the median proves to be a better measure than the mean. Income is

the most common example of a distribution that is typically skewed. Because of the disproportionate weight of a few top-salary jobs, the arithmetic mean for income is nearly always artificially inflated. For income, the median is a good choice because it is not affected by extreme values.

■ EXAMPLE 1

Five individuals working for a small firm have annual incomes of $30,000, $45,000, $45,000, $45,000, and $200,000. Find the median, mode, and mean.

The *median* is $45,000 because it is the middle observation. The *mode* is the most common observation: $45,000. The *mean* is

$$\frac{30,000 + (3)45,000 + 200,000}{5} = \frac{365,000}{5} = \$73,000$$

which does not match any of the salaries. ■

Suppose an emergency stock clerk who handles different sizes of crutches wants to know which is the most popular size (the mode) so that he can order enough to meet his demand. By looking at the several measures of central tendency, he can obtain some idea of the shape of the frequency distribution. In a symmetrical distribution (Figure 4.1), the three measures of central tendency are identical. In an asymmetrical distribution (see Figure 4.2), the mode remains located (by definition) at the peak; the mean is off to the right; and the median is in between. Left-skewed distributions are the mirror image of Figure 4.2.

Generally, modes are used for nominal scores, medians for ordinal scores, and means for interval scores.

Mean
Median
Mode
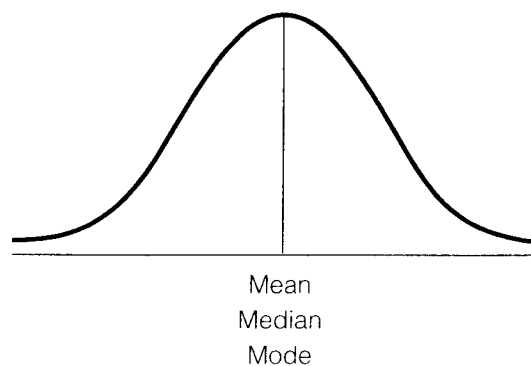
Figure 4.1   Symmetrical Frequency Distribution
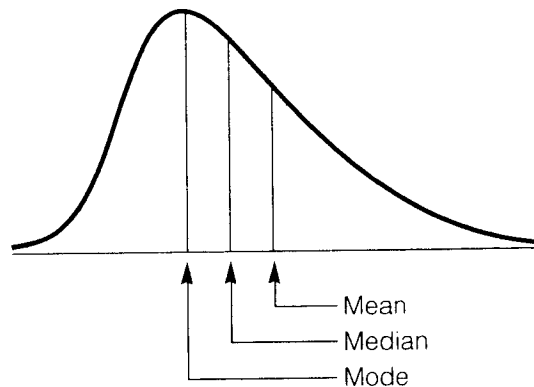
Figure 4.2 Asymmetrical Distribution,
Skewed to the Right

4.2   MEASURES OF VARIATION

Knowing a distribution's central tendency is helpful, but it is not enough. It is also important to know whether the observations tend to be quite similar (homogeneous) or whether they vary considerably (heterogeneous). To describe variability, measures of **variation** have been devised. The most common of these are the *range*, the *mean deviation*, and the *standard deviation*.

Range

The **range** is defined as the difference in value between the highest (maximum) and lowest (minimum) observation:

$$\text{Range} = x_{max} - x_{min} \tag{4.3}$$

The range can be computed quickly, but is not very useful because it considers only the extremes and does not take into consideration the bulk of the observations.

Mean Deviation

By knowing the range of a data set, we can gain some idea of the set's variability. The **mean deviation** is a bit more sophisticated than the range. It is defined as the average deviation of all observations from the mean. We can compute how far observations deviate from the mean by subtracting the mean from the value of each observation. The mean deviation is the sum of all the **absolute values** of the deviations divided by the number of observations—that is,

$$\text{Mean deviation} = \frac{|x_1 - \bar{x}| + |x_2 - x| + \cdots + |x_n - x|}{n} \tag{4.4}$$

Table 4.1 Annual Percentage of Medical School National Board Honorees, 1988–1992

| | Year of graduation | | | | |
	1988	1989	1990	1991	1992
Percent of honors graduates (x_i)	4	6	5	8	7
Deviation from mean $(x_i - x)$	-2	0	-1	2	1
Absolute value of deviation from mean $(\mid x_i - x \mid)$	2	0	1	2	1
Squared deviation from mean $(x_i - x)^2$	4	0	1	4	1

where $\mid x_1 - \bar{x} \mid$ is read as "the absolute value of x sub one minus x-bar." Absolute value ignores the sign of the difference; that is, the mean deviation indicates how much, on average, the observations deviate from the arithmetic mean. The mean deviation is now mainly of historical interest; the measure was more commonly used before the age of electronic calculators and computers.

As an example, consider the percentage of graduates of a medical school who passed their National Boards with honors during a five-year period (Table 4.1). Note that some of the deviations are positive, some are negative, and one is zero. In sum, because $\bar{x}$ is the balance point of the observations, they add to zero. By using absolute values, we can eliminate the negative signs and thus compute a mean deviation:

$$\text{Mean deviation} = \frac{\Sigma \mid x - \bar{x} \mid}{n} = \frac{6}{5} = 1.20$$

For the five-year period, the percentage of graduates earning honors differed, on average, by 1.2 percentage points from the mean of 6%.

Standard Deviation

By far the most widely used measure of variation is the **standard deviation,** represented by the symbol s. It is the square root of the *variance* of the observations. The **variance,** or s^2, is computed by squaring each deviation from the mean, adding them up, and dividing their sum by one less than n, the sample size:

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1} \tag{4.5}$$

The sample variance may thus be thought of as the mean squared deviation from the mean, and the greater the deviations, the greater the variance.

The variance is readily computed for the data of Table 4.1 as follows:

$$x = \frac{30}{5} = 6$$

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - x)^2}{n - 1}$$

$$s^2 = \frac{4 + 0 + 1 + 4 + 1}{4} = \frac{10}{4} = 2.5$$

The standard deviation is computed by extracting the square root of the variance. Or symbolically,

$$s = \sqrt{s^2} \tag{4.6}$$

For our example, $s = \sqrt{2.5} = 1.58$. (The square root of a number is best obtained with a calculator or from square root tables found as an appendix in many statistics books.) The value $s = 1.58$ indicates that, on the average, observations fall 1.58 units from the mean. Equation 4.6 and the mathematically equivalent calculating equation are summarized in Table 4.2.

==Both the variance and the standard deviation are measures of variation in a set of data==. The larger they are, the more heterogeneous the distribution. For example, if we were to compare the National Board scores of graduates of two medical schools, the school with the smaller standard deviation would have students who are more homogeneous in ability than the school with the larger. That is, the school with the smaller s will have scores closer to the mean and the school with the larger s will have scores scattered over a wider range around the mean.

Table 4.2 Equations for Means and Standard Deviations

	Definition Equation		Calculating Equation
	Ungrouped Data		
Mean	$x = \dfrac{\sum_{i=1}^{n} x_i}{n}$ (4.2)		Same
Standard deviation	$s = \sqrt{\dfrac{\sum_{i=1}^{n} (x_i - x)^2}{n - 1}}$	[(4.5) and (4.6)]	$\sqrt{\dfrac{\sum_{i=1}^{n} x_i^2 - \dfrac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n - 1}}$ (4.8)

Frequently, the symbol SD is used to denote the Standard Deviation, s, which is usually obtained from s^2. However, SD^2 can also be used to calculate the value of the sample variance if SD is known.

As a measure of variation, standard deviation is much preferred over all other choices. The units of the standard deviation turn out to be the same as the units of the raw data (e.g., inches, millimeters, kilograms), whereas the units of variance are squared. Standard deviation is arithmetically easy to handle and avoids the awkwardness of absolute values. Because the magnitude of the standard deviation depends on the phenomenon being observed, which may be represented by large or small numbers, the standard deviation itself can be large or small. What is a large deviation for one variable may be small for another.

Understanding the sources of variation may help you appreciate the meaning of standard deviation. For example, among subjects, one source of variation may be due to a personal characteristic such as age or sex. Another source may be individual variation; still another, the varying condition of the subject (i.e., observations obtained before or after dinner, or before or after exercise, may differ). Yet another source of variation is measurement error. Although a certain amount is inherent in any observation, scientists strive mightily to keep it to a minimum by use of appropriate experimental designs.

COEFFICIENT OF VARIATION

One important application of the mean and the standard deviation is the **coefficient of variation.** It is defined as the ratio of the standard deviation to the absolute value of the mean, expressed as a percentage.

$$CV = \frac{100s}{|\bar{x}|}\% \tag{4.7}$$

The coefficient of variation depicts the size of the standard deviation relative to its mean. Because both standard deviation and the mean represent the same units, the units cancel out and the coefficient of variation becomes a pure number; that is, it is free of the measurement units of the original data. Therefore, it is possible to use it to compare the relative variation of even unrelated quantities. For example, we may wish to know whether the variation of blood glucose readings is greater or less than the variation of serum cholesterol levels. From Table 3.1, we can compute the variation exactly. The coefficient of variation for blood glucose (in milligrams per deciliter) is $54.72/152.14 \times 100 = 36\%$; and for serum cholesterol it is $38.82/216.96 \times 100 = 18\%$. From this we see that the variation in blood glucose is relatively greater than that in serum cholesterol.

.4   MEANS AND STANDARD DEVIATIONS OF A POPULATION

The equations given for the mean and the standard deviation apply to the data of a sample selected from a population. When we have data for an *entire* population, we use similar equations but different symbols. Table 4.3 compares equations used for the two purposes. The **population mean,** μ (lowercase Greek mu), is defined as the sum of the values divided by N, the number of observations for the entire population. The sample mean, $\bar{x}$, is an estimate of μ and is the sum of the values in the sample divided by n, the number of observations in the sample alone. (Convention dictates the use of Greek letters for population parameters and Roman letters for sample statistics.) The population variance, σ^2, is the sum of the squared deviations from the population mean μ divided by N, whereas the sample variance s^2 (an estimation of σ^2) is the sum of the squared deviations from the sample mean $\bar{x}$ divided by $n - 1$. Dividing by $n - 1$ looks like a peculiarity, but it provides an equation that gives an unbiased sample variance; that is, the mean of all possible samples of a particular sample size gives the correct answer if $n - 1$ is used as a divisor. Therefore, the use of $n - 1$, instead of n, gives a more accurate estimate of σ^2. In Chapter 2, we gave the definitions for a parameter and a statistic. These can now be illustrated with the mean and standard deviation. Because both μ and σ are characteristics of a population, they are parameters. And because $\bar{x}$ and s are characteristics of a sample, they are statistics. Convention dictates the use of some shorthand to replace more awkward notation. Hence, in subsequent chapters, we will use x instead of x_i and Σ instead of

$$\sum_{i=1}^{n}$$

Conclusion

In describing data by use of a summary measure, it is important to select the measure of central tendency that best represents the data accurately. A better

Table 4.3 Equations for Population and Sample Means and Standard Deviations

Quantity	Sample		Population	
Mean	$x = \dfrac{\sum_{i=1}^{n} x_i}{n}$	(4.2)	$\mu = \dfrac{\sum_{i=1}^{N} x_i}{N}$	(4.11)
Variance	$s^2 = \dfrac{\sum_{i=1}^{n} (x_i - x)^2}{n - 1}$	(4.5)	$\sigma^2 = \dfrac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}$	(4.12)
Standard deviation	$s = \sqrt{s^2}$	(4.6)	$\sigma = \sqrt{\sigma^2}$	(4.13)

way of representing data is to use two summary measures—one to indicate central tendency and one to indicate variation. The most commonly used pair is the arithmetic mean and the standard deviation.

Vocabulary List

absolute values	mean deviation	range
central tendency	median	standard deviation
coefficient of variation	mode	variance
mean	population mean	variation

Exercises

4.1   Find the mean, median, mode, range, variance, and standard deviation for the data 8, 5, 1, 5, 2, 3. (For variance, use equation 4.5.)

4.2   Using the sample 3, 4, 6, 1, 10, 6,
a. find the median, mean, and range
b. compute the standard deviation using equations 4.5 and 4.6
c. compute the standard deviation using equation 4.8
d. compare the results of (b) and (c)

Why is the standard deviation of this example larger than that of Exercise 4.1?

4.3   Determine the range, median, and mode for the data of Table 2.2.

4.4   Assuming that Table 2.2 is a population of values, compute the mean, variance, and standard deviation. (Use the equation $\sigma^2 = \Sigma x^2/N - \mu^2$ for the calculation of variance.)

4.5   Compute $\bar{x}$, s^2, and s for the sample of 10 that you took in Exercise 2.1. (Use equation 4.8.) Compare your results with those for Exercise 4.4.

4.6   Determine the mean, variance, and standard deviation of weights in Table 3.1 by using the equations of Table 4.2.

4.7   a. Calculate the coefficient of variation for the heights and weights given in Table 3.1. (Use the results from Exercise 4.6.)
b. Compare the two coefficients. Which one is larger? Approximately how many times larger?

4.8   a. Calculate the mean and the standard deviation for the systolic blood pressure values given in Table 3.1. (*Hint:* Use the equations of Table 4.2.)
b. Calculate $\bar{x} - s$ and $\bar{x} + s$.
c. Calculate $x - 2s$ and $\bar{x} + 2s$.
d. Calculate $x - 3s$ and $\bar{x} + 3s$.
e. What percentage of the blood pressure observations fall within each of the three intervals you calculated in (b), (c), and (d)?

4.9   a. Find the median age of the sample represented in Table 3.1.
b. What is the age range?

4.10   For the cholesterol values given in Table 3.1, the mean and the standard deviation are, respectively, 216.96 and 38.82. What is the variance?

4.11   If the variance of blood glucose values in Table 3.1 is 2994, what is the standard deviation?

4.12   List some practical uses for standard deviation.

4.13   Describe a situation in which it would be useful to know
a. the mean, median, and mode
b. primarily the median
c. primarily the mean

4.14   a. Refer to Table 3.1. Using equation 4.8, calculate the mean and the standard deviation of systolic blood pressure
    i. for those who have had no education (code = 1)
    ii. for those who have had intermediate education (code = 3)
b. Compare the standard deviations of the two groups. Which set of values has the larger standard deviation and by how much?
c. From your computations in (b), draw a conclusion about the relative variation of the observations in the two groups.

4.15   Define
a. measure of central tendency
b. mean
c. median
d. mode
e. population mean
f. sample standard deviation
g. population variance
h. range
i. deviation
j. coefficient of variation

4.16   Explain what happens to the mean, median, and standard deviation if 10, the fifth observation, is replaced by 2 in Exercise 4.2.

4.17   Explain what these symbols and formulae mean
a. Σx
b. $(\Sigma x)^2$
c. Σx^2
Is $(\Sigma x)^2$ always larger than Σx^2?

4.18   Using the results of Exercise 4.8(a) and Exercise 4.10,
a. compute the coefficient of variation for the systolic blood pressure values
b. compute the coefficient of variation for the cholesterol values
c. compare the two coefficients. What are their units?

4.19   What would you consider to be the major distinction between a population variance and a sample variance?

4.20   Describe the characteristics of a frequency distribution if
a. $\bar{x} = 15$ and the median is 19

b. $x = 19$ and the median is 15

c. $\bar{x} = 17$ and the median is 17

4.21 a. Why is the standard deviation rather than the variance used more commonly to describe the spread of a distribution?

b. Why is the sum of the deviations $[\Sigma(x_i - \bar{x})]$ always zero?

c. Explain how it is possible for a person to drown in a river whose average depth is 12 inches.

d. How would you explain the sentence "The average American is a 33-year-old white woman"?

4.22 Using the sample values: 1, 2, 3, 4, 4, 5, 6,

a. find the mean, median, and mode

b. find the standard deviation

c. find the coefficient of variation

4.23 What is the standard deviation for a data set that has a mean of 16 and a variance of 144.

4.24 What would be the mean and standard deviation if in Exercise 4.22

a. each observation is increased by two units?

b. each observation is multiplied by a factor of 2?

4.25 Describe the frequency distribution

a. in which the $\bar{x}$ = median = mode

b. if the median = 10, the mode = 5, and $\bar{x}$ = 15

4.26 Explain the basic difference in the formulas of $\bar{x}$ and μ.

4.27 Calculate the mean, median, mode and standard deviation for *each* of these distributions.

A (2, 3, 4, 4, 4, 5, 6)

B (2, 3, 4, 4, 4, 5, 20)

C (−5, −4, −3, 0, 3, 4, 5)

a. Which measure of central tendency would be the "best" or most useful measure for each group? Briefly justify your choice.

b. Which distribution is skewed?

4.28 If there is a large numerical difference between the mean and median the distribution is probably_____?

4.29 If you have one or more extreme scores in a data set, which measure of central tendency is most likely to be affected?

4.30 Identify the measure of central tendency that would be most appropriate for the following data sets:

a. prices of homes in a community

b. ages of incoming freshmen

c. number of apples per tree in a commercial orchard

d. blood pressure readings of college students

5 Probability

Chapter Outline

5.1 What Is Probability?
Discusses the concept of probability as a measure of the likelihood of occurrence of a particular event

5.2 Complementary Events
Demonstrates how to calculate probability when events are complementary

5.3 Probability Rules
Solves problems involving the probability of compound events by use of the addition rule or the multiplication rule, or conditional probability

5.4 Counting Rules
Explains how to compute the number of possible ways an event can occur by use of permutations and combinations

5.5 Probability Distributions
Illustrates the concept of a probability distribution, which lists the probabilities associated with the various outcomes of a variable

5.6 Binomial Distribution
Describes a common distribution having only two possible outcomes on each trial

Learning Objectives

After studying this chapter, you should be able to

1. State the meaning of "probability" and compute it in a given situation
2. State the basic properties of probability
3. Select and apply the appropriate probability rule for a given situation
4. Distinguish between mutually exclusive events and independent events
5. Distinguish between permutations and combinations; be able to compute them for various events
6. Explain what a probability distribution is and state its major use

7. State the properties of a binomial distribution

8. Compute probabilities by using a binomial distribution

9. Interpret the symbols in the binomial term

WHAT IS PROBABILITY?

A pregnant woman wonders about the chance of having a boy or a girl baby. An understanding of probability can throw some light on this question. Any answer must be based on various assumptions. If she assumes that bearing a boy or a girl is equally likely, she will expect one boy baby for every two births—that is, half the time. As another way of estimating her chances of having a boy, she could count the number of boys and girls born in the past year. Vital statistics indicate there are about 1056 live births of boys for every 1000 live births of girls, so she could estimate her probability of having a boy as

$$\frac{1056}{2056} = .514$$

It should be noted that the term *probability* applies exclusively to a future event, never to a past event (even if its outcome is unknown). Therefore it is really not appropriate to state that the woman's probability of bearing a boy is .514, because, upon conception, the sex of the fetus is already established. It would be more appropriate to discuss the probability *before* the baby is conceived.

Many events in life are inherently uncertain. Probability may be used to measure the uncertainty of the outcome of such events. For example, you may wish to learn the probability of survival to age 80, of developing cancer, or of becoming divorced. This chapter attempts to cover some of the basic concepts of probability and set forth some rules and models that, if followed, can provide some quantitative estimates of the occurrence of various events.

Probability statements are numeric, defined in the range of 0 to 1, never more and never less. A probability of 1.0 means that the event will happen with certainty; 0 means that the event will not happen. If the probability is .5, the event should occur once in every two attempts on the average. If the probability is close to 1.0, then the event is more likely to happen, and if the probability is close to 0, it is unlikely to happen.

There are many ways of defining probability. Here is one of the simplest definitions: **Probability** is the ratio of the number of ways the specified event can occur to the total number of **equally likely events** that can occur. This definition was implicit in our example of estimating a woman's probability of bearing a boy baby.

The probability of an event, $P(E)$, can be defined as the proportion of times a favorable event will occur in a long series of repeated trials:

$$P(E) = \frac{n}{N} = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} \tag{5.1}$$

■ EXAMPLE 1

One coin: In a toss of a fair coin, there are two possible outcomes, a head (H) or a tail (T); that is, $N = 2$. So the probability of having a head equals

$$P(H) = \frac{1}{2} \quad ■$$

Note: The word *fair* implies that the coin or dice are not loaded; that is, they will give a fair representation to each outcome in a large number of tosses.

■ EXAMPLE 2

Two coins: In a toss of two coins, four outcomes are possible: HT, TH, TT, HH. (HT means heads on the first coin and tails on the second.) There are two helpful ways to ensure that all possible outcomes are listed—the **tree diagram** (Figure 5.1) and the **contingency table** (Table 5.1).

Consider the following questions: What is the probability of flipping two heads? At least one head? No heads? One head and one tail? Not more than one
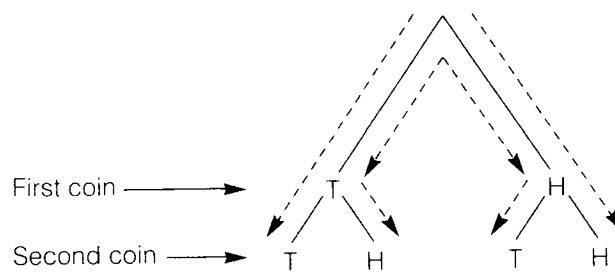
Figure 5.1   A Tree Diagram

Table 5.1 A Contingency Table

		Second Coin	
		H	T
First coin	H	HH	HT
	T	TH	TT

tail? We can tabulate the answers as follows:

Probability of an Event	Favorable Events
$P(2H) = \dfrac{1}{4}$	HH
$P(\text{at least } 1H) = \dfrac{3}{4}$	HT, TH, HH
$P(0H) = \dfrac{1}{4}$	TT
$P(1H \text{ and } 1T) = \dfrac{2}{4}$	HT, TH
$P(\text{not more than } 1T) = \dfrac{3}{4}$	HT, TH, HH

■

■ **EXAMPLE 3**

Dice: In a roll of a fair die, there are six equally possible outcomes ($N = 6$): 1, 2, 3, 4, 5, and 6. You might ask, "What is the probability of rolling a particular number?" And the answer is

$$P(\text{even number}) = \frac{3}{6}$$

$$P(2 \text{ or } 3) = \frac{2}{6}$$

$$P(\text{greater than } 3) = \frac{3}{6} \quad ■$$

Mutually exclusive events, E_i, are events that cannot happen simultaneously; that is, if one event happens, the other event cannot happen. Thus in the one-coin example, E_1 (heads) and E_2 (tails) are mutually exclusive, and their probabilities add up to 1.

Denoted symbolically, the three basic properties of probability for mutually exclusive events are

$$0 \leq P(E_i) \leq 1 \tag{5.2}$$

$$P(E_1) + P(E_2) + \cdots + P(E_n) = 1 \tag{5.3}$$

$$P(\text{not } E_1) = 1 - P(E_1) \tag{5.4}$$

where $E_1, E_2, \ldots, E_n$ are mutually exclusive outcomes.

By perusing our three examples, you can see that (1) the probability of an event is always between 0 and 1 (inclusive); it is never negative and never greater than 1; and (2) the sum of the probabilities of all mutually exclusive out-

comes is equal to 1; and (3) the probability of an event E_1 not occurring is equal to 1 less the probability of E_1.

2 COMPLEMENTARY EVENTS

The complementary event A is the complement of event A as shown in Figure 5.2. We observe that

$P(A) = $ sum of probabilities of outcomes in A +

$P(\bar{A}) = $ sum of probabilities of outcomes in $\bar{A} = 1$

and Probability of $(\bar{A})$ is

$P(A) + P(\bar{A}) = 1$

Therefore,

$P(\bar{A}) = 1 - P(A)$
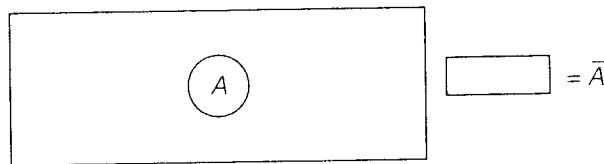
Figure 5.2   Complement of Event A

.3 PROBABILITY RULES

Two indispensable rules help answer the most common questions concerning the probability of compound events (those composed of two or more individual events). These are the *multiplication rule* and the *addition rule*.

Multiplication Rule

Two events are **independent** if the occurrence of one has no effect on the chance of occurrence of the other. The outcomes of repeated tosses of a coin illustrate independent events, for the outcome of one toss does not affect the outcome of any future toss. Note that "independent" and "mutually exclusive" are not the same. The occurrence of one independent event does not affect the chance of another such event occurring at the same time, whereas mutually exclusive events cannot occur simultaneously.

To determine the probability of occurrence of two independent events, we use the multiplication rule. The **multiplication rule** states that the probability of occurrence of two independent events, A and B, is equal to the product of the probabilities of the individual events.

Symbolically,

$$P(A \text{ and } B) = P(A)P(B) \tag{5.5}$$

■ **EXAMPLE 4**

In tossing two coins, what is the probability that a head will occur both on the first coin (H_1) *and* on the second coin (H_2)? The solution:

$$P(H_1 \text{ and } H_2) = [P(H_1)][P(H_2)] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} \quad ■$$

■ **EXAMPLE 5**

Suppose the probability that a typical driver will have an accident during a given year is $\frac{1}{10}$. What is the probability that two randomly selected drivers will *both* have an accident during the year? The solution:

$$P = \left(\frac{1}{10}\right)\left(\frac{1}{10}\right) = \frac{1}{100} \quad ■$$

Conditional Probability Calculating the probability of an event using, in the denominator, a *subset* of all possible outcomes will give a *conditional probability*. As we will see from Example 6, the probability of stopping smoking during pregnancy is

$$\frac{768}{4075} = .188$$

However, the probability of stopping smoking during pregnancy given that the subgroup consists of those who have 16 years of education is

$$\frac{214}{884} = .242$$

The .188 is the value of a simple probability and .242 is the value of a conditional probability. **Conditional probability** is denoted by $P(A \mid B)$. It is the probability that A occurs, given that B has occurred, and is given by the following ratio:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \qquad \text{providing } P(B) \text{ is not equal to zero} \qquad (5.6)$$

The vertical line in $P(A \mid B)$ is read "given."

■ EXAMPLE 6

From the data on stopping smoking during pregnancy given in Table 5.2, we can calculate several probabilities. For example, if A is the event of stopping smoking during pregnancy and B is the event that mothers have 16 years of education, then

$$P(A) = \frac{768}{4075} = .188$$

is the probability of selecting a mother who has stopped smoking. The probability of selecting a woman who has 16 years of education is

$$P(B) = \frac{884}{4075} = .2169$$

and the probability of selecting a mother who has both stopped smoking and has 16 years of education is

$$P(A \wedge B) = \frac{214}{4075} = .0525$$

The conditional probability of stopping smoking during pregnancy given that the mother has 16 years of education can be obtained using the formula

Table 5.2 Number of Mothers of Live-Born Infants Who Stopped Smoking During Pregnancy by Educational Status

| Smoking Status | Years of Education (%) | | | | | | | | | |
	0–11 yrs	%	12 yrs	%	13–15 yrs	%	16 yrs	%	Total	%
Stopped	42	9.7	308	16.9	204	21.8	214	24.2	768	18.8
Did not	390	90.3	1515	83.1	732	78.2	670	75.8	3307	81.2
Total	432	100.0	1823	100.0	936	100.0	884	100.0	4075	100.0

Source: U.S. National Natality Survey, 1980.

$$P(A \mid B) = \frac{P(A \wedge B)}{P(B)} = \frac{.0525}{.2169} = .242$$

Note that the probability obtained using the last formula, $P(A \mid B) = .242$, is the same as that obtained directly from the frequencies in Table 5.2, namely,

$$\frac{214}{884} = .242 \quad \blacksquare$$

Let us consider the difference between $P(A)$ and $P(A \mid B)$. $P(A)$ gives the probability that event A occurs out of all possible outcomes, whereas $P(A \mid B)$ gives the probability that event A occurs given that we restrict ourselves to a subset of all possible B outcomes. These two probabilities are not the same unless the two events are independent. The general rule that permits us to make such a statement is

Events A and B are *independent* if $P(A \mid B) = P(A)$

From Example 6 we can see that events A and B are not independent because $P(A \mid B) = .242$ does not equal $P(A) = .1885$. A modification of Example 3 illustrates how we can check if events A and B are independent. If A is the event of an even number on the toss of a fair die and B is the event that we consider only the first four numbers, then the two events A and B are independent because their probabilities are equal:

$$P(A) = \frac{3}{6} = \frac{1}{2} \quad \text{and} \quad P(A \mid B) = \frac{2}{4} = \frac{1}{2}$$

Addition Rule

To determine the probability that one or another event (but not necessarily both) will occur, we use the addition rule. The **addition rule** states that the probability that event A or event B (or both) will occur equals the sum of the probabilities of each individual event less the probability of both. Symbolically,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \tag{5.7}$$

The reason for subtracting $P(A$ and $B)$ is that this portion would otherwise be included twice, as you can see from Figure 5.3a, which is an example of a **Venn diagram.** In such a diagram, circles within a rectangular space represent events; and the relationship between those events is indicated by a separation or an intersection of the circles. The area excluding A is denoted with a bar over it, $\bar{A}$. The area of not A or not B is denoted as $\bar{A}\,\bar{B}$.

(a) Nonmutually exclusive events

(b) Mutually exclusive events
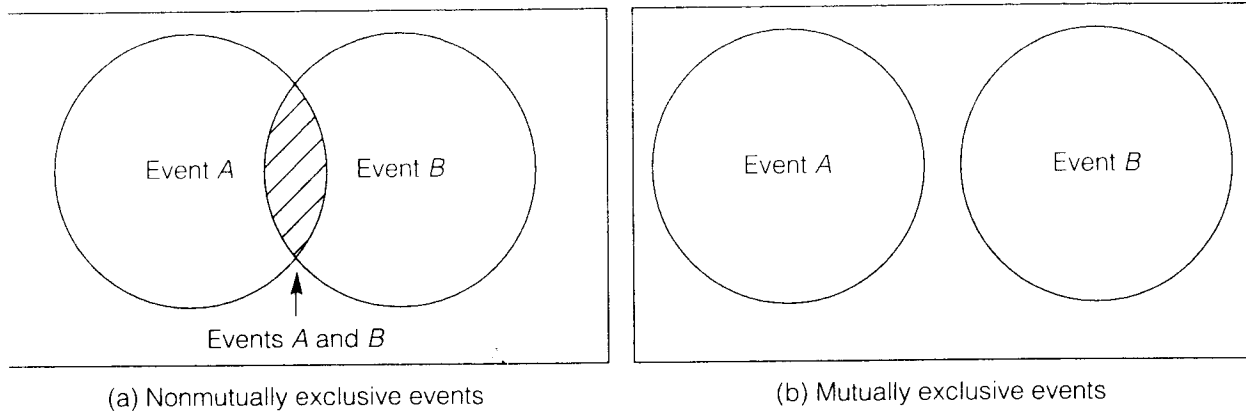
igure 5.3   Venn Diagrams of Two Events

■ EXAMPLE 7

In flipping two coins, you may wish to know the probability of having a head on the first coin (H_1), or on the second (H_2), *or* on both (H_1H_2). To get the answer, use the addition rule:

$$P(H_1 \text{ or } H_2) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4} \quad ■$$

■ EXAMPLE 8

What is the probability that you will obtain a 3 *or* 4 on one toss of a die? The addition rule gives

$$P(3 \text{ or } 4) = P(3) + P(4) - P(3 \text{ and } 4) = \frac{1}{6} + \frac{1}{6} - 0 = \frac{1}{3} \quad ■$$

Recall that whenever two events are *mutually exclusive*, the probability of both events occurring is equal to zero. By tossing a 3, you have *excluded* the probability of tossing a 4. Likewise, you cannot simultaneously flip a head and a tail with a coin. Hence, the addition rule is somewhat simplified when the two events are mutually exclusive. The rule then becomes

$$P(A \text{ or } B \text{ or both}) = P(A) + P(B) \tag{5.8}$$

The $P(A \text{ and } B)$ term of equation 5.7 is zero; it drops out (Figure 5.3b).

■ EXAMPLE 9

At birth, the probability that a U.S. female will survive to age 65 is approximately $\frac{8}{10}$; that is, $P(F_{65}) = \frac{8}{10}$. The probability that a male will survive to age 65 is

approximately $\frac{2}{3}$; that is, $P(M_{65}) = \frac{2}{3}$. What is the probability that a U.S. female will die before age 65? Using equation 5.4, we see that the probability of dying before age 65, $P(F_d)$, is computed by subtracting from 1 the probability of surviving to age 65:

$$P(F_d) = 1 - P(F_{65}) = 1 - \frac{8}{10} = .2$$

Carrying the example further, the following probabilities can be computed by appropriately applying the multiplication and addition rules:

1. The probability that both will be alive at age 65:

$$P = P(M_{65})P(F_{65}) = \left(\frac{2}{3}\right)\left(\frac{8}{10}\right) = .533$$

2. The probability that only the male will be alive at age 65:

$$P = P(M_{65})P(F_d) = \frac{2}{3}\left(1 - \frac{8}{10}\right)$$

$$= .133$$

3. The probability that only the female will be alive at age 65:

$$P = P(F_{65} \text{ and } M_d) = P(F_{65})P(M_d) = \frac{8}{10}\left(1 - \frac{2}{3}\right) = .267$$

4. The probability that at least one of the two will be alive at age 65:

$$P = P(\text{either one or both will be alive})$$

$$= P(F_{65} \text{ and } M_{65}) + P(M_{65} \text{ and } F_d) + P(F_{65} \text{ and } M_d)$$

$$= .533 + .133 + .267 = .933$$

This answer may also be obtained by finding the probability of the complement of both the male and the female dying; that is,

$$1 - P(M_d \text{ and } F_d) = 1 - \frac{1}{3} \cdot \frac{2}{10} = .933 \blacksquare$$

.4   COUNTING RULES

In computing the probabilities of various events, we first need to know in how many possible ways such events can occur. For example, if we wish to know the probability of having two girls and a boy in a three-child family, it is essential to know the order of their birth. How many different possibilities are there of having two girls and a boy? The number of different outcomes is eight:

girl girl girl   boy girl girl*

girl girl boy*   boy girl boy

girl boy girl*   boy boy girl

girl boy boy   boy boy boy

Here you can see that the three outcomes marked with asterisks qualify as successes (two girls and a boy).

You may need to know the number of different possibilities of a certain event in order to determine the denominator you need to use to compute a probability. Three general rules are helpful in obtaining counts.

Rule 1: Number of Ways

If an event A can occur in n_1 distinct ways and event B can occur in n_2 ways, then the events consisting of A and B can occur in $n_1 \cdot n_2$ ways.

■ EXAMPLE 10

If you had three different diet (D) choices by amount of protein (low, medium, high) and three different choices by amount of fat (low, medium, high), there would be $(n_1)(n_2) = (3)(3) = 9$ different possible diets:

D_1: protein (low), fat (low) D_4: protein (low), fat (medium)

D_2: protein (medium), fat (low) D_5: protein (medium), fat (medium)

D_3: protein (high), fat (low) D_6: protein (high), fat (medium)

          D_7: protein (low), fat (high)

          D_8: protein (medium), fat (high)

          D_9: protein (high), fat (high) ■

Rule 2: Permutations

In determining the number of ways in which you can manage a group of objects, you must first know whether the *order* of arrangement plays a role. For example, the order of arrangement of a person's missing teeth is important, but

the order of selecting a group for a committee is not, because any order results in the same committee.

A **permutation** is a selection of r objects from a group of n objects, taking the order of selection into account. The number of different ways in which n objects may be arranged is given by n!. The exclamation mark stands for **factorial**, and the symbol n! (read "n factorial") means $n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1$. Thus 3! (i.e., three factorial) $= 3 \cdot 2 \cdot 1 = 6$, and 0! $= 1$. This last, 0! $= 1$ may seem arbitrary, but it is a mathematically necessary convention that keeps us from dividing by zero.

■ **EXAMPLE 11**

If we wish to identify vials of a medication by using three different symbols, x, y, and z, how many different ways can the vials be identified? The answer is

$$3! = 3 \cdot 2 \cdot 1 = 6$$

The six different identifications are xyz, xzy, yxz, yzx, zxy, and zyx. ■

Suppose we want to learn the number of ways of selecting r objects from a set of n objects and *order is important.* Here we would use the equation

$$P(n,r) = \frac{n!}{(n - r)!} \tag{5.9}$$

■ **EXAMPLE 12**

If there are three effective ways of treating a cancer patient—surgery (S), radiation (R), and chemotherapy (C)—in how many different ways can a patient be treated with two different treatments if the order of treatment is important? The answer is given by

$$P(3,2) = \frac{3!}{(3 - 2)!} = \frac{3 \cdot 2 \cdot 1}{1} = 6$$

or SR, RS, CS, SC, RC, and CR. ■

Rule 3: Combinations

Sometimes we may wish to determine the number of arrangements of a group of objects when order is not important, as in selecting books from a shelf. A **combination** is a selection of a subgroup of distinct objects, with order not being important. The equation for obtaining the number of ways of selecting r objects from n objects, disregarding order, is

$$C(n,r) = \frac{n!}{r!(n - r)!}$$ (5.10)

where C denotes the total number of combinations of objects.

■ **EXAMPLE 13**

Suppose that three patients with snakebites are brought to a physician. To his regret, he discovers that he has only two doses of antivenin. The three patients are a pregnant woman (w), a young child (c), and an elderly man (m). Before deciding which two to treat, he examines his choices:

$$C(3,2) = \frac{3!}{2!(3 - 2)!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} = 3$$

The three choices are wc, wm, cm. Note that cw, mw, and mc are the same as the first three because order does not matter. ■

5.5   PROBABILITY DISTRIBUTIONS

A key application of probability to statistics is estimating the probabilities that are associated with the occurrence of different events. For example, we may wish to know the probability of having a family of two girls and one boy or the probability that two out of three patients will be cured by a certain medication. If we know the various probabilities associated with different outcomes of a given phenomenon, we can determine which outcomes are common and which are not. This helps us reach a decision as to whether certain events are significant. A complete list of all possible outcomes, together with the probability of each, constitutes a **probability distribution.**

The outcome of events may be described numerically (e.g., the number of three-boy families). The symbol X usually denotes the variable of interest. This variable, which can assume any number of values, is called a **random variable** because it represents a chance (random) outcome of an experiment. Thus, we can say that a probability distribution is a list of the probabilities associated with the values of the random variable obtained in an experiment. Random variables may be either discrete or continuous. Only discrete variables are discussed in this chapter.

Three examples of probability distribution are illustrated in Table 5.3. As the third example in the table shows, if a family is selected at random, the probability that it is a three-boy family is .125. In this example, the number of boys is the random variable.

From the distributions in Table 5.3, we can again see that the sum of the probabilities of a set of mutually exclusive events always equals 1.

Table 5.3 Examples of Probability Distribution

Toss of Two Coins		Roll of a Die		Sex of Three-Child Family	
E	$P(E)$	E	$P(E)$	E	$P(E)$
HH	$\frac{1}{4}$	1	$\frac{1}{6}$	3 boys*	.125
HT	$\frac{1}{4}$	2	$\frac{1}{6}$	2 boys, 1 girl	.375
TH	$\frac{1}{4}$	3	$\frac{1}{6}$	1 boy, 2 girls	.375
TT	$\frac{1}{4}$	4	$\frac{1}{6}$	3 girls	.125
	1.0	5	$\frac{1}{6}$		1.000
		6	$\frac{6}{6}$		
			1.0		

*For ease of computation, we assume that P(boy) = .5.

BINOMIAL DISTRIBUTION

In practice, we usually work with distributions that are reasonable approximations to theoretical distributions. In constructing a frequency table, we can obtain an estimate of the probability distribution by visualizing the relative frequency associated with each possible outcome. Having this information, we can make statements about how common any given event is.

Various phenomena follow certain underlying mathematical distributions. One of the most useful, the **binomial distribution,** serves as a model for outcomes limited to two choices—sick or well, dead or alive, at risk or not at risk. For such a dichotomous population, we may wish to know the probability of having a number of r successes on n different attempts, where the probability of success on any one attempt is p.

As an example, let's again consider the probability that a couple planning three children will have two girls and one boy. Suppose we wonder whether the three children will arrive in the sequence GGB. If we assume that the probability of having a girl is .5, then the probability of the sequence GGB occurring is $(\frac{1}{2} \cdot \frac{1}{2}) \cdot \frac{1}{2} = \frac{1}{8}$. However, two girls and a boy may arrive in three different ways—GGB, GBG, BGG—as indicated by $C(3,2) = 3$, where $C(3,2)$ denotes the combination of three things taken two at a time. Since the probability of each sequence is $\frac{1}{8}$, the probability of having two girls and a boy in *any* sequence is

$$3 \cdot \binom{1}{8} = 3(.125) = .375$$

as indicated in the third probability distribution in Table 5.3.

The probability distribution for this example is algebraically obtained from the expansion of the **binomial term** $(p + q)^n$, where p is the probability of a successful outcome, $q = 1 - p$ is the probability of an unsuccessful outcome, and n is the number of trials or attempts. The binomial expansion is applicable, providing that

1. Each trial has only two possible outcomes—success or failure
2. The outcome of each trial is independent of the outcomes of any other trial
3. The probability of success, p, is constant from trial to trial

Under these conditions, the probability of the sequence GGB is

$$p \cdot p(1 - p) = p^2 q$$
$$\text{G} \quad \text{G} \quad \text{B}$$

and the probability of any sequence of two girls and a boy is

$$C(3,2)p^2(q) = \frac{3!}{2!(3 - 2)!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^3 = .375$$

where $C(3,2)$ becomes the binomial coefficient giving the number of different sequences of three children consisting of two girls and one boy.

In general, the probability of an event consisting of r successes out of n trials is

$$P(r \text{ successes}) = \frac{n!}{r!(n - r)!}p^r q^{n-r} \tag{5.11}$$

where n = the number of trials in an experiment
r = the number of successes
$n - r$ = the number of failures
p = the probability of success
$q = 1 - p$, the probability of failure

The expression

$$\frac{n!}{r!(n - r)!}p^r q^{n-r}$$
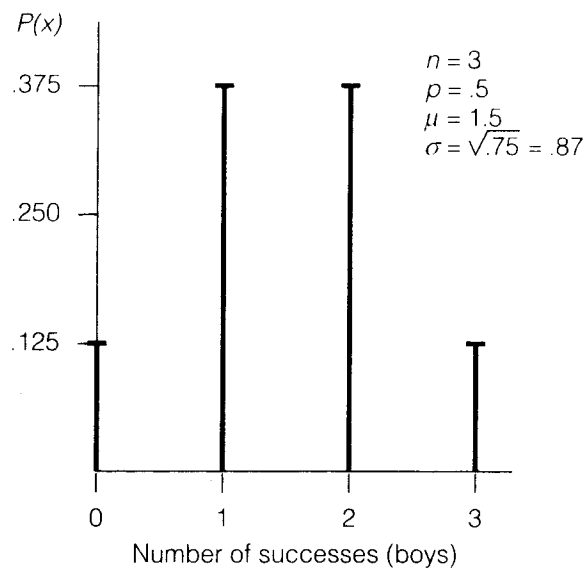
Figure 5.4   Example of Binomial Distribution

is a term from the binomial expansion. The entire expansion lists the terms for r successes and $(n - r)$ failures from the binomial distribution:

$$(p + q)^3 = q^3 + \frac{3!}{1!(3 - 1)!}pq^2 + \frac{3!}{2!(3 - 2)!}p^2q + p^3$$

$$= q^3 + 3pq^2 + 3p^2q + p^3 \tag{5.12}$$

$$= P(3F) + P(1S, 2F) + P(2S, 1F) + P(3S)$$

where F = failure, and S = success. If a "success" means bearing a girl ($p = .5$), equation 5.12 reduces to

$$(p + q)^3 = \left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^3$$

$$= .125 + .375 + .375 + .125 = 1.000 \tag{5.13}$$

$$= P(3B) + P(1G, 2B) + P(2G, 1B) + P(3G)$$

Equation 5.13 shows that the binomial expansion yields the binomial distribution illustrated initially in the third example in Table 5.3 and visually portrayed in Figure 5.4.

It is essential that you gain a feeling for the meaning of a binomial term, so that you will then be able to construct or interpret one for any occasion. Figure 5.5 should enable you to understand the anatomy of the binomial term. Note especially that in a binomial distribution, r (the number of favorable outcomes) serves as the random variable. Using the probability distribution given in equation 5.11, you can find the following probabilities in a three-child family:

$$\frac{n!}{r!(n-r)!}$$

Number of ways an event can occur

$$p^r \quad q^{(n-r)}$$

Probability of an event with r successes and $n-r$ failures
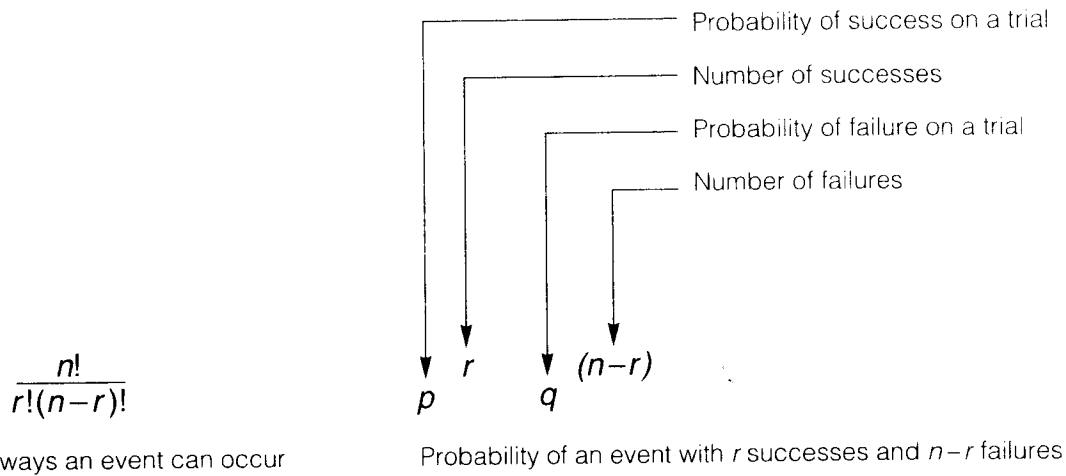
Figure 5.5 Identification of the Components of the Binomial Term

3B	$= .125$
2G, 1B	$= .375$

At most 2G (3B; 2B, 1G; 1B, 2G) $= .125 + .375 + .375 = .875$

At least 1B (3B; 2B, 1G; 1B, 2G) $= .125 + .375 + .375 = .875$

The probabilities of a binomial term can be obtained by reading them directly from the binomial probability table found in Appendix A. A small portion of this table is reproduced in Table 5.4.

■ **EXAMPLE 14**

What is the probability of having two girls and one boy in a three-child family if the probability of having a boy is .5?
From the calculations in equation 5.13, we can see that

$$P(2G, 1B) = \frac{3!}{2!(3 - 2)!}\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = 3(.125) = .375$$

Looking at Table 5.4 with $n = 3$, $p = .5$, and $r = 2$, we again find that $P = .375$.
■

Table 5.4 Portion of Binomial Probability Table

				P	
n	r	.10	.25	1/3	.50
	0	.7290	.4219	.2963	.1250
	1	.2430	.4219	.4444	.3750
3	2	.0270	.1406	.2222	.3750
	3	.0010	.0156	.0370	.1250

The binomial expansion is used to obtain the probability of various events when n is small, say 30 or less. When n is large, you should see the Gaussian (normal) distribution, discussed in the next chapter, as an approximation. To do this, you need to know the mean and the standard deviation of the *binomial* distribution, and we will consider these in Chapter 11.

Conclusion

Probability measures the likelihood that a particular event will or will not occur. In a long series of trials, probability is the ratio of the number of favorable outcomes to the total number of equally likely outcomes. Permutations and combinations are useful in determining the number of outcomes. If compound events are involved, we need to select and apply the addition rule or the multiplication rule to compute probabilities. The outcome of an experiment, together with its respective probabilities, constitutes a probability distribution. One very common probability distribution is the binomial. It presents the probabilities of various numbers of successes in trials where there are only two possible outcomes to each trial.

Vocabulary List

addition rule	equally likely events	permutation
binomial distribution	factorial	probability
binomial term	independent events	probability distribution
combination	multiplication rule	random variable
conditional probability	mutually exclusive	tree diagram
contingency table	events	Venn diagram

Exercises

5.1 Two coins are tossed and the results observed. Find the probabilities of observing zero heads, one head, two heads.

5.2 Take two coins, toss them 20 times, and record the number of heads observed for each toss. Compute the proportion of zero heads, one head, and two heads, and compare the results with the expected results you computed in Exercise 5.1.

5.3 A fair coin is tossed three times and the number of heads observed. Determine the probability of observing
a. exactly two heads
b. at least two heads
c. at most two heads
d. exactly three heads

5.4 A couple is planning to have three children. Find the following probabilities by listing all the possibilities and using equation 5.1:
a. two boys and one girl
b. at least one boy

    c. no girls

    d. at most two girls

    e. two boys followed by a girl

    How does (e) differ from (a)?

5.5   Suppose you observe the result of a throw of a single fair die. How many times would you expect to observe a 1 in 60 throws? How many times would you expect to observe each of the other possibilities (2, 3, 4, 5, 6) in 60 throws?

5.6   Toss a die 60 times and record the frequency of occurrence of 1, 2, 3, 4, 5, 6. Compare your results with those in Exercise 5.5. In your judgment, is the die you tossed a fair one? (You will learn in Chapter 12 how to apply a statistical test to determine the fairness of a die.)

5.7   On a single toss of a pair of fair dice, what is the probability that

    a. a sum of 8 is observed?

    b. a sum of 7 or 11 comes up?

    c. a sum of 8 or a double appears?

    d. a sum of 7 appears and both dice show a number less than 4?

5.8   A ball is drawn at random from a box containing 10 red, 30 white, 20 blue, and 15 orange balls. Find the probability that it is

    a. orange or red

    b. neither red nor blue

    c. not blue

    d. white

    e. red or white or blue

5.9   In an experiment involving a toxic substance, the probability that a white mouse will be alive for 10 hours is 7/10, and the probability that a black mouse will be alive for 10 hours is 9/10. Find the probability that, at the end of 10 hours,

    a. both mice will be alive

    b. only the black mouse will be alive

    c. only the white mouse will be alive

    d. at least one mouse will be alive

5.10   If an individual were chosen at random from Table 2.2, what is the probability that that person would be

    a. a vegetarian?

    b. a female?

    c. a male vegetarian?

5.11   Suppose a person is randomly selected from Table 3.1. Find the probability that he or she

    a. has completed high school or technical school

    b. is a smoker

    c. is physically inactive (code number = 1)

    d. is a physically inactive smoker

    e. has a serum cholesterol greater than 250 and a systolic blood pressure above 130

    f. has a blood glucose level of 100 or less

5.12   In how many ways can five differently colored marbles be arranged in a row?

5.13 In how many ways can a roster of 4 club officers be selected from 10 nominees so that the first one selected will be president; the second, vice-president; the third, secretary; and the fourth, treasurer?

5.14 Compute
a. $P(8,3)$
b. $P(6,4)$

5.15 In how many ways can a committee of five people be chosen out of nine people?

5.16 Calculate
a. $C(7,4)$
b. $C(6,4)$
Compare (b) with Exercise 5.14b. What do you observe?

5.17 In how many ways can 10 objects be split into two groups containing 4 and 6 objects respectively?

5.18 About 50% of all persons three years of age and older wear glasses or contact lenses. For a randomly selected group of five people and using equation 5.11, compute the probability that
a. exactly three wear glasses or contact lenses
b. at least one wears them
c. at most one wears them

5.19 If 25% of 11-year-old children have no decayed, missing, or filled (DMF) teeth, find the probability that in a sample of 20 children there will be
a. exactly 3 with no DMF teeth
b. 3 or more with no DMF teeth
c. fewer than 3 with no DMF teeth
d. exactly 5 with no DMF teeth
(*Hint:* Refer to the first example in Table 5.3.)

5.20 It is known that approximately 10% of the population is hospitalized at least once during a year. If 10 people in such a community are to be interviewed, what is the probability that you will find
a. all have been hospitalized at least once during the year?
b. 50% have been hospitalized?
c. at least 3 have been hospitalized?
d. exactly 3 have been hospitalized?
(*Hint:* Refer to the first example in Table 5.3.)

5.21 Seventy-five percent of youths 12–17 years of age have a systolic blood pressure less than 136 mm of mercury. What is the probability that a sample of 12 youths of that age group will include
a. exactly 4 who have a systolic pressure greater than 136?
b. no more than 4 who have a blood pressure greater than 136?
c. at least 4 who have a blood pressure greater than 136?
(*Hint:* Refer to the first example in Table 5.3.)

5.22 Assuming that, of all persons 17 years and over, half the males and one-third of the females are classified as presently smoking cigarettes, find the probability that in a randomly selected group of 10 males and 15 females

a. exactly 10 smoke (4 males, 6 females)
b. all smoke
c. none smoke
(*Hint:* Refer to the first example in Table 5.3.)

5.23 Define the following:
a. equally likely events
b. mutually exclusive events
c. independent events
d. probability
e. conditional probability
f. probability distribution
g. random variable

5.24 Define and give an example of
a. combination
b. permutation
c. factorial
d. addition rule
e. multiplication rule

5.25 Using the data from Table 5.2, let the event

A = a mother with less than 12 years of education

B = a mother who has quit smoking

a. Calculate $P(A)$.
b. Calculate $P(B)$.
c. Calculate $P(B \mid A)$.
d. Indicate whether events A and B are independent. (*Hint:* Use equations 5.5 and 5.6.)

5.26 a. Define the binomial distribution.
b. Define the components of a binomial term.

5.27 Using the data from Table 3.2 and Table 3.3, prepare a new frequency table of systolic blood pressure for nonsmokers and smokers. Using this new table, let the events

A = a nonsmoker

B = a smoker

C = a systolic blood pressure of 170 or greater

Find
a. $P(A)$
b. $P(B)$
c. $P(C)$
d. $P(C \mid A)$
e. $P(C \mid B)$
Compare (d) and (e) and comment. Are smoking status and blood pressure level independent?

5.28 Use Table 5.2 to compute some probabilities you could use in persuading some-one that the level of education and smoking are inversely related.

5.29 Phenylketonuria (PKU) is a genetic disease that occurs if the person inherits two recessive genes (meaning that this person has the inability to metabolize the amino acid, phenylalanine, into another amino acid, tyrosine). The possible genetic combinations are: two dominant genes (no disease), one dominant and one recessive gene (no disease, but a carrier), and two recessive genes (have PKU). Calculate the probability of a child having the disease if
  a. both parents are carriers
  b. one parent is a carrier, the other has two dominant genes
  c. one parent has the disease, the other has two dominant genes
  d. both parents have the disease

5.30 Using the information from Exercise 5.29, calculate the probability of a child being a carrier if
  a. both parents are carriers
  b. one parent is a carrier, the other has two dominant genes
  c. one parent has the disease, the other has two dominant genes
  d. both parents have the disease

5.31 Using the information from Exercise 5.29, calculate the probability of a child having two dominant genes if
  a. both parents are carriers
  b. one parent is a carrier, the other has two dominant genes
  c. one parent has the disease, the other has two dominant genes
  d. both parents have the disease

5.32 Sickle cell anemia is a genetic disease that occurs if the person inherits two recessive genes. The possible genetic combinations are: two dominant genes (no disease), one dominant and one recessive gene (has sickle cell trait, which means the person is a carrier) and two recessive genes (have sickle cell anemia). Calculate the probability of a child having the disease if
  a. both parents have sickle cell trait
  b. one parent is a carrier, the other has two dominant genes
  c. one parent has the disease, the other has two dominant genes
  d. both parents have the disease

5.33 Using the information from Exercise 5.32, calculate the probability of a child being a carrier if
  a. both parents have sickle cell trait
  b. one parent is a carrier, the other has two dominant genes
  c. one parent has the disease, the other has two dominant genes
  d. both parents have the disease

5.34 Using the information from Exercise 5.32, calculate the probability of a child having two dominant genes if
  a. both parents have sickle cell trait
  b. one parent is a carrier, the other has two dominant genes
  c. one parent has the disease, the other has two dominant genes
  d. both parents have the disease

6 The Normal Distribution

Learning Objectives

After studying this chapter, you should be able to

1. State why the normal distribution is so important
2. Identify the properties of the normal distribution
3. Interpret the mean and the standard deviation in the context of the normal curve
4. List the differences between the normal and the standard normal distribution
5. Explain the standard normal score $Z = (x - \mu)/\sigma$
6. Compute the percentage of areas between given points under a normal curve
7. Compute percentiles of specified variables by using a table of standard normal scores

6.1 THE IMPORTANCE OF NORMAL DISTRIBUTION

Physicians often rely on a knowledge of **normal limits** to classify patients as healthy or otherwise. For example, a serum cholesterol level above 200 mg/dl is widely regarded as indicating a significantly increased risk for coronary heart disease. An accurate determination of such a value, whether or not based on a mathematical model, is of critical importance. The decision may be a matter of

life or death, because the physician uses the findings to decide what type of treatment to prescribe for a patient. It would be unfortunate, perhaps tragic, if the "normal limits" were faulty. In that case, some patients might receive an unnecessary treatment, while others might fail to receive a needed treatment.

Serum albumin is the chief protein of blood plasma. For any group of persons, the concentrations of serum albumin tend to follow a **normal distribution.** The normal limits for albumin are calculated by adding and subtracting 2 standard deviations from the mean of a large set of observations obtained from a group of presumably healthy persons. This calculation provides the limits that contain the middle 95% (the "normal range") of observations but exclude the remaining 5%, of which 2.5% falls in the lower tail and 2.5% in the upper tail. Extreme observations, those in the tails, are considered unusual and may be regarded as presumptive evidence of a health problem. However, not all variables follow a normal distribution. Two well-known counterexamples are urea and alkaline phosphatase. For these, use of the same method would give incorrect "normal limits" that would not include 2.5% of the observations in each tail. In response to this problem, medical statisticians Elveback, Guillier, and Keating (1970) have suggested that "clinical limits" rather than "normal limits" be used. **Clinical limits** are the lower and upper 2.5 percentage points for any distribution, normal or otherwise, of healthy persons. Clinical limits are obtained empirically, not by adding and subtracting 2 standard deviations from the mean. Use of clinical limits is greatly preferred to use of normal limits, because the term *normal limits* has been grossly misused and fallen into disrepute.

In Chapter 5, we learned how a distribution of a variable gives an idea of the values of its population. Knowing that a variable is distributed normally can be especially helpful in drawing inferences as to how frequently certain observations are likely to occur.

The normal distribution, perhaps the most important of statistical distributions, was first discovered by the French mathematician Abraham Demoivre in 1733, and rediscovered and applied to the natural and social sciences by the French mathematician Pierre Simon de Laplace and the German mathematician and astronomer Karl Friedrich Gauss in the early nineteenth century. Sir Francis Galton, a cousin of Charles Darwin, first applied the normal curve to medicine.

Scholars like to refer to the normal curve as the **Gaussian distribution.** This preference is in reaction to a tendency of some persons to feel that anything not "normally" distributed is "abnormal." However, in popular practice, most statisticians and scientists still call it the normal distribution.

There are a legion of reasons why the normal distribution plays such a key role in statistics. For one thing, countless phenomena follow (or closely approximate) the normal distribution. Just a few of them are height, serum cholesterol, life span of light bulbs, body temperature of healthy persons, size of oranges, brightness of galaxies. But there are likewise countless phenomena that do *not* follow the normal distribution, ranging from individual annual income to

clinical laboratory readings for urea, magnesium, or alkaline phosphatase. Another reason for the normal distribution's popularity is that it possesses certain mathematical properties that make it attractive and easy to manipulate. Still another reason is that much statistical theory and methodology was developed around the assumption that certain data are distributed approximately normally. Normal distribution is the basis for the use of inferential statistics.

6.2   PROPERTIES OF THE NORMAL DISTRIBUTION

The normal distribution has three main properties. First, it has the appearance of a symmetrical **bell-shaped curve** extending infinitely in both directions. It is symmetrical about the mean μ. Not every bell-shaped curve, however, is a normal distribution.

Second, all normal distributions have a particular internal distribution for the area under the curve. Whether the mean or standard deviation is large or small, the relative area between any two designated points is always the same. Let us look at three commonly used points along the abscissa. In Figure 6.1 we see that 68.26% of the area is contained within $\mu \pm 1\sigma$, 95.45% within $\mu \pm 2\sigma$, and 99.74% within $\mu \pm 3\sigma$ (see Table A, inside back cover).
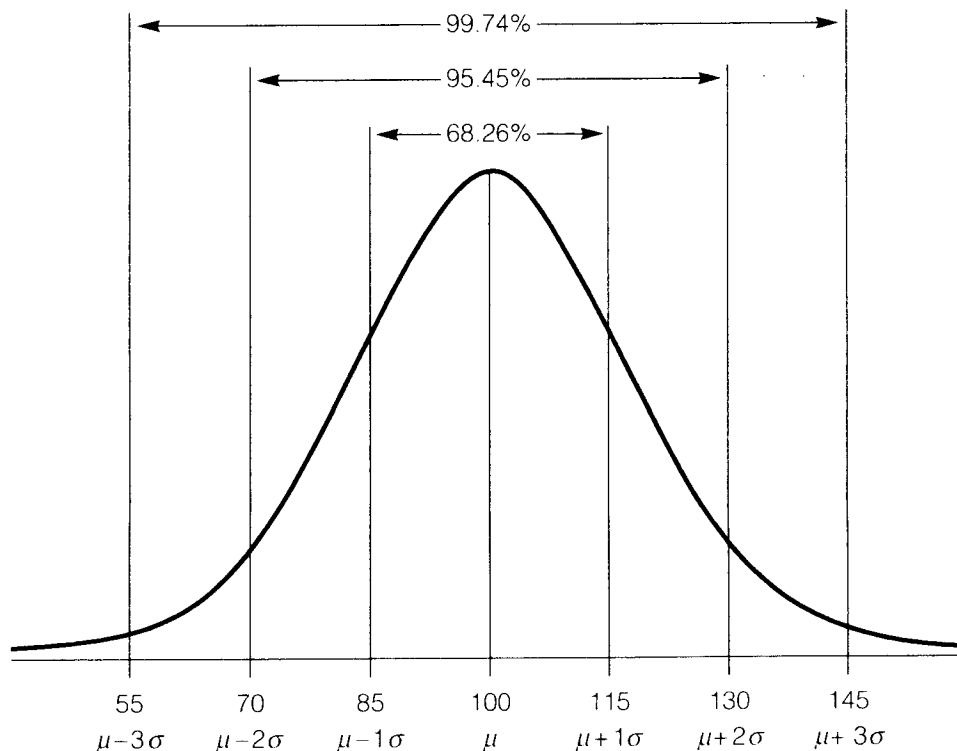
Figure 6.1   Important Divisions of the Normal Distribution of IQs

The amount of area under the normal curve is directly proportional to the percentage of raw scores. For example, if you have .20 of the total area of 1.0, you have .20 or 20% of the raw scores. The total area under the curve in Figure 6.1 equals 1.0. This is a nice feature. Because of it, the area under the curve between any two points can be interpreted as the relative frequency (or probability of occurrence) of the values included between those points.

Third, the normal distribution is a theoretical distribution defined by two parameters: the mean μ and the standard deviation σ. The **exponential equation** for the normal distribution is

$$y = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2 \right] \tag{6.1}$$

where y is the height of the curve for a given value x, exp is the base of the natural logarithms (approximately 2.71828), and π is the well-known constant (about 3.141519).

AREAS UNDER THE NORMAL CURVE

Let us assume that the IQ of a given population is normally distributed with $\mu = 100$ and $\sigma = 15$. In that case, 68.3% of the IQ scores (rounded up) should fall between 85 and 115 (100 ± 15), as shown in Figure 6.1. Similarly, we would expect approximately 95% of the IQs to fall between 70 and 130, 2.5% above 130, and 2.5% below 70. To find the proportion of persons with IQ scores between 130 and 135, we need a table of normal curve areas. But first, let us see how to use such a table.

Because it would be out of the question to tabulate the areas of all possible normal curves, we use the feature that all normal curves are symmetrical and have an area of 1.0. Thus, dealing with one normal curve is like dealing with any other, provided we use a standardized unit. Such a unit is the **standardized score,** Z, which gives the relative position of any observation in the distribution. If a variable is normally distributed, then *any* individual raw score can be converted into a corresponding Z score. Sometimes Z is referred to as **Z score, Z value,** or **standard normal score.** Thus, for the normal curve, the Z score is obtained by

$$Z = \frac{x - \mu}{\sigma} \tag{6.2}$$

Standardized observations provide an indication as to how many standard deviations an observation falls either below or above the mean. You can appreciate the effect of this transformation on the mean and the standard deviation of x by following a few simple steps:

	Variable	Mean	Standard Deviation
Step 1: Start with x	x	μ	σ
Step 2: Subtract μ	$x - \mu$	$\mu - \mu = 0$	σ
Step 3: Divide by σ	$Z = \frac{1}{\sigma}(x - \mu) = \frac{x - \mu}{\sigma}$	$\left(\frac{1}{\sigma}\right)0 = 0$	$\left(\frac{1}{\sigma}\right)\sigma = 1$

In step 1, given the variable x, the mean is μ and the standard deviation is σ. In step 2, on subtraction of μ, the mean is shifted from μ to 0, but σ is left unchanged. In step 3, the variable is divided by σ, the mean remains 0, and σ reduces to 1.

The net effect of this so-called Z transformation is to change any normal distribution to the **standard normal distribution,** where $\mu = 0$ and $\sigma = 1$. An example of this transformation is how IQ scores are established. The population is defined as all those who take the test. All of the scores are tabulated and a population mean and population standard deviation are calculated. The mean is given a score of 100 or the 50th percentile. Each standard deviation is established at 15 points. This example will be used throughout the chapter to explain how the normal distribution works.

It is this distribution that takes on prominence because of its use in setting confidence limits and tests of hypotheses. Areas for the standard normal distribution are listed in Table A (inside back cover). Here are a few pointers for anyone using it for the first time. Figure 6.2 shows areas under the standard normal curve between various points along the abscissa. The proper use of Table A may
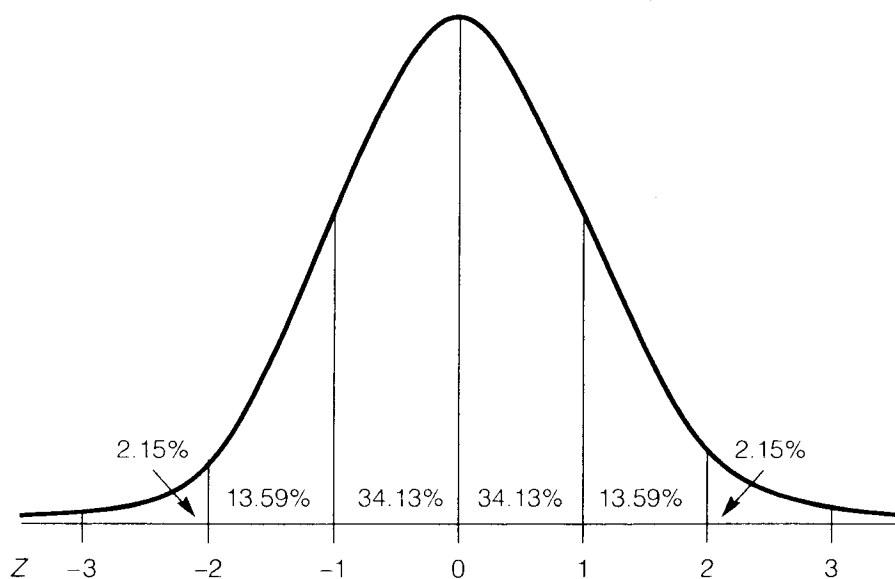
Figure 6.2 Areas Under the Standard Normal Curve

be demonstrated by finding the areas between different points along the abscissa. The area under the curve, A, is tabulated in the body of the table; it is that area between zero and some point Z to the right of zero. Z values are given in the left margin. Whole numbers and tenths are read at the left; hundredths along the top, horizontally. Further, since the normal curve is symmetrical, the area between zero and any negative point is equal to the area between zero and the corresponding positive point. Remember that because the area under the curve is equal to 1 and the curve is symmetrical about zero, the area to the right of Z can be computed by subtracting from .5. (Another way of explaining this is that area A (between the mean and Z) plus area B (Z and beyond) *always* equals .5.)

Now let us extend our IQ score example to illustrate various uses of Table A.

■ **EXAMPLE 1**

What is the proportion of persons having IQs between 100 and 120?

Sketch a curve like the one in Figure 6.3. Shade in the area you wish to find. Transform the IQ variable to a Z score. The Z corresponding to $x = 100$ is

$$Z = \frac{x - \mu}{\sigma} = \frac{100 - 100}{15} = 0$$

and the Z corresponding to $x = 120$ is

$$Z = \frac{120 - 100}{15} = \frac{20}{15} = 1.33$$

By using Table A to find the area for a Z of 1.33, you will find the answer to be .4082. Therefore, the proportion of persons having IQs between 100 and 120 is .4082, about 41%. ■
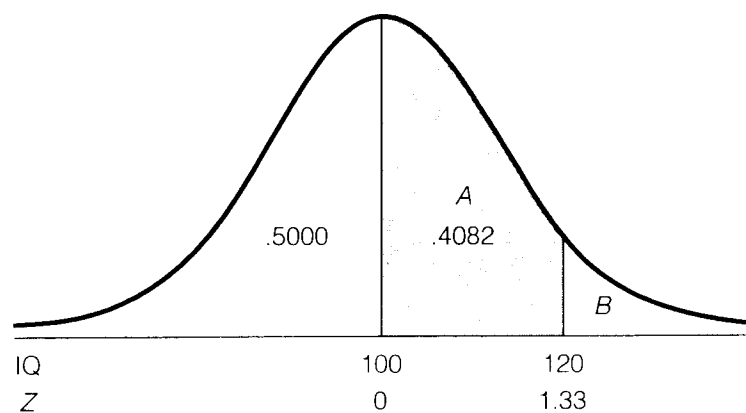
Figure 6.3 Area Corresponding to IQs Between 100 and 120

■ EXAMPLE 2

What proportion of persons has IQs greater than 120?

Again, sketch a curve, this time following the model of Figure 6.4. Because the area to the right of $Z = 0$ is .50, and the area between $Z = 0$ and 1.33 is 0.4082, by subtraction you will obtain the area beyond Z of 1.33, namely, $.5000 - .4082 = .0918$. So the answer is that about 9% have IQs over 120. ■
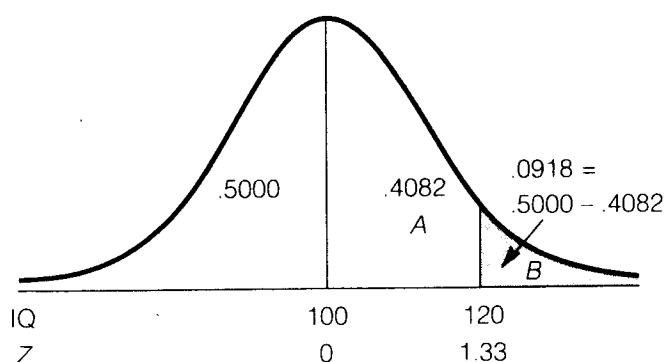
Figure 6.4   Area Corresponding to IQs Above 120

■ EXAMPLE 3

What is the proportion of persons with IQs between 80 and 120?

That is the same as asking what proportion is found under the normal curve between the standardized values of Z between -1.33 and $+1.33$. Using the symmetry argument, you simply double the area between $Z = 0$ and 1.33, namely, $2(.4082) = .8164$; that is, 82% have IQs between 80 and 120. Figure 6.5 illustrates this solution. ■

We should point out that a $-Z$ score means that the corresponding raw score will be lower than the mean. In this example, a raw score of 80 corresponds to a
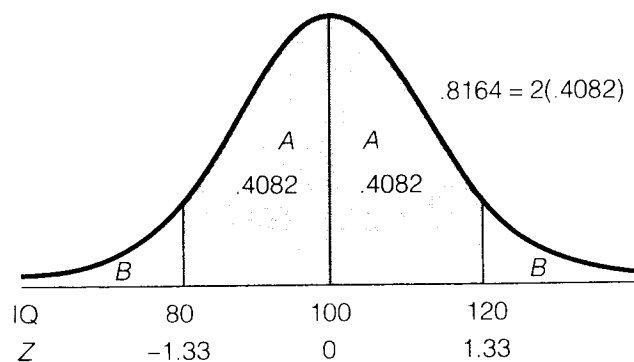
Figure 6.5   Area Corresponding to IQs Between 80 and 120

$-Z$ score of -1.33. Notice that the area between the mean and Z (both labeled area A) are exactly the same; the only difference is that the positive Z score represents the area above the mean and the $-Z$ represents the area below the mean. A Z score of -1.33 means that a raw score of 80 is 1.33 standard deviations below the mean and a Z score of 1.33 means that a raw score of 120 is 1.33 standard deviations above the mean.

■ **EXAMPLE 4**

What is the proportion of persons with IQs between 95 and 125?
The corresponding Z scores for two areas, A_1 and A_2, are

$$A_1: Z = \frac{95 - 100}{15} = \frac{-5}{15} = -.33$$

$$A_2: Z = \frac{125 - 100}{15} = \frac{25}{15} = 1.67$$

Figure 6.6 illustrates the two areas.
The area (A_1) between $Z = 0$ and $-.33$ is the same, of course, as that between 0 and $+.33$. In Table A, we see that A_1 is .1293 and that A_2, between $Z = 0$ and 1.67, is .4525. Thus $A_1 + A_2 = .1293 + .4525 = .5818$. The answer, then, is that about 58% of this population have IQs between 95 and 125. ■
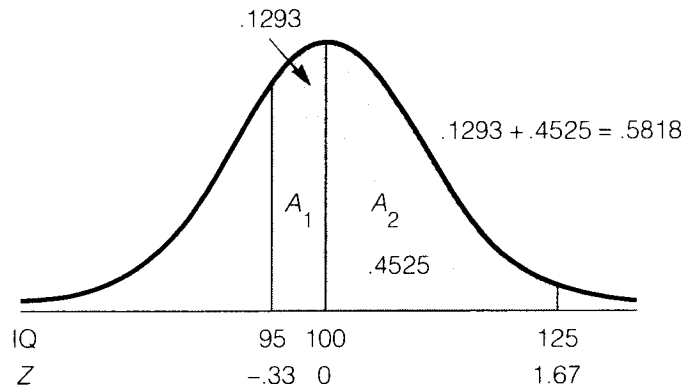
Figure 6.6   Area Corresponding to IQs Between 95 and 125

Table A may also be used to determine the Z value that corresponds to any given area, as, for instance, the upper 10% of the curve. Consequently, we can obtain the value on the abscissa that corresponds to the 90th **percentile**, P_{90}.

■ **EXAMPLE 5**

What is the Z value of the normal curve that marks the upper 10% (or 90th percentile) of the area?
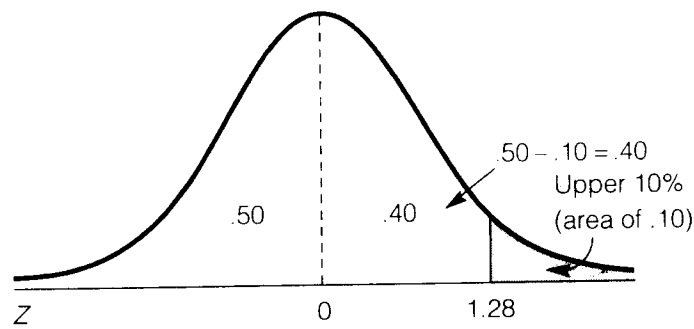
Figure 6.7   Normal Deviate Corresponding to the Upper 10% of IQs

The desired Z score is that value corresponding to .40 of the area, as Figure 6.7 illustrates. In Table A the value is found to be approximately $Z = 1.28$. ∎

■ EXAMPLE 6

What is the 90th percentile of IQ scores?

This is the logical extension of Example 5. We just found the Z of the 90th percentile to be 1.28. But what does this mean in terms of IQs? The answer is found by a simple application of equation 6.2.

$$Z = \frac{x - \mu}{\sigma}$$

$$1.28 = \frac{x - 100}{15}$$
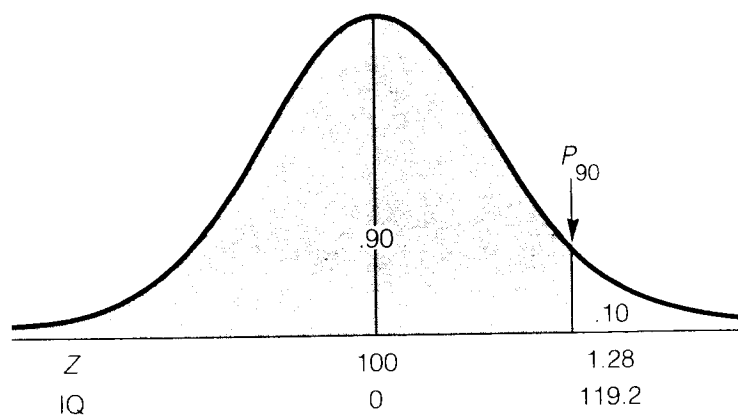
Figure 6.8   90th Percentile of the Distribution of IQs

Therefore

$$x = \mu + Z\sigma \quad \text{(formula)}$$

$$x = 1.28(15) + 100 = 119.2 \quad \text{(example)}$$

Thus 119.2 is the 90th percentile of IQ scores, as illustrated in Figure 6.8. ■

Knowing how to compute areas under a normal curve makes it easy to find the proportion (*probability*) of persons possessing certain cholesterol values, heights, or any other variable that is normally distributed. Knowing the probability of a given event allows us to draw appropriate inferences as to the expected occurrence of that event.

Conclusion

The normal distribution is an important concept for a number of reasons. It has been used to define "normal limits" for clinical variables. Many variables follow a normal distribution. The assumption of normality proves extremely useful because of the exceptional properties of the distribution. We can quickly reduce any normal distribution to the standard normal distribution by transforming the variable to a normal deviation Z score. Because these Z scores and the normal curve areas corresponding to them are conveniently tabulated, we are able to compute the probability of occurrence of various events and thus to decide about the degree of uniqueness of those events.

Vocabulary List

bell-shaped curve
clinical limits
exponential equation
normal distribution
  (Gaussian
  distribution)

normal limits
percentile
standard normal
  distribution

standarized score
Z score (Z value;
  standard normal
  score)

Exercises

6.1   Find the areas under the normal curve that lie between the given values of Z.
a. $Z = 0$ and $Z = 2.37$
b. $Z = 0$ and $Z = -1.94$
c. $Z = -1.85$ and $Z = 1.85$
d. $Z = -0.76$ and $Z = 1.13$
e. $Z = 0$ and $Z = 3.09$
f. $Z = -2.77$ and $Z = -0.96$

6.2   Determine the areas under the normal curve falling to the right of Z (or to the left of $-Z$).

a. $Z = 1.73$
b. $Z = -2.41$ and $Z = 2.41$
c. $Z = 2.55$
d. $Z = -3$ and $Z = 3$
e. $Z = 5$

6.3   What Z scores correspond to the following areas under the normal curve?
a. area of .05 to the right of $+Z$
b. area of .01 to the left of $-Z$
c. area of .05 beyond $\pm Z$
d. area of .01 beyond $\pm Z$
e. area of .90 between $\pm Z$
f. area of .95 between $\pm Z$

6.4   Find the standard normal score for
a. the 95th percentile
b. the 80th percentile
c. the 50th percentile

6.5   The accompanying figure shows the assumed distribution for systolic blood pressure readings of a large male population.
a. Determine the normal deviates Z for the various cutoff points.
b. Find the equivalent cutoff points in terms of systolic blood pressures if the mean reading is 130 and the standard deviation is 17.

| 5% | 5% | 40% | 40% | 5% | 5% |

Z

Hypotensive   Borderline   Normal   Borderline   Hypertensive
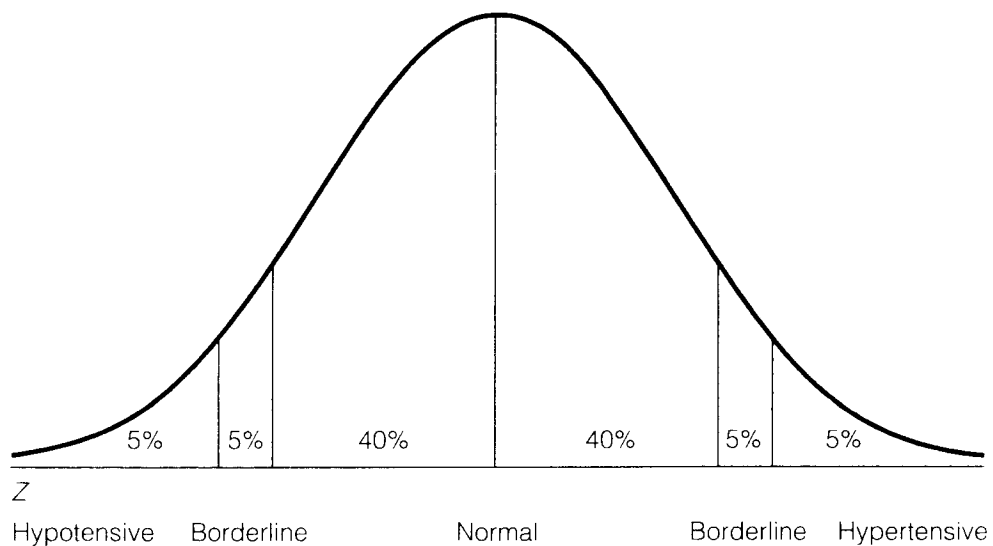
6.6   If the heights of male youngsters are normally distributed with a mean of 60 inches and a standard deviation of 10, what percentage of the boys' heights (in inches) would we expect to be
a. between 45 and 75?
b. between 30 and 90?
c. less than 50?

    d. 45 or more?
    e. 75 or more?
    f. between 50 and 75?

6.7 For Exercise 6.5, find the 95th percentile.

6.8 An instructor is administering a final examination. She tells her class that she will give an A to the 10% of the students who earn the highest grades. Past experience with the same examination has shown that the mean grade is 75 and the standard deviation is 8. If the present class runs true to form, what grade would a student need in order to earn an A?

6.9 Assume that the age at onset of disease X is distributed normally with a mean of 50 years and a standard deviation of 12 years. What is the probability that an individual afflicted with X had developed it before age 35?

6.10 a. What is the distinction between a normal distribution and the standard normal distribution?
    b. Why do statisticians prefer to work with the standard normal distribution rather than the normal distribution?

6.11 a. Describe the normal distribution.
    b. Give two examples of random variables that appear to be normally distributed.
    c. What is the probability that the value of a randomly selected variable from a normal distribution will be more than 3 standard deviations from its mean value?

6.12 a. Suppose that 25-year-old males have a remaining mean life expectancy of 55 with a standard deviation of 6. What proportion of 25-year-old males will live past 65?
    b. What assumption do you have to make in (a) to obtain a valid answer?

6.13 Explain why "clinical limits" may be more appropriate than "normal limits" in classifying certain clinical values as "abnormal."

6.14 If IQ scores are normally distributed with $\mu = 100$ and $\sigma = 15$, what is the probability of a randomly selected subject with an IQ between 100 and 133?

    .4861

    100   133
     0   2.2
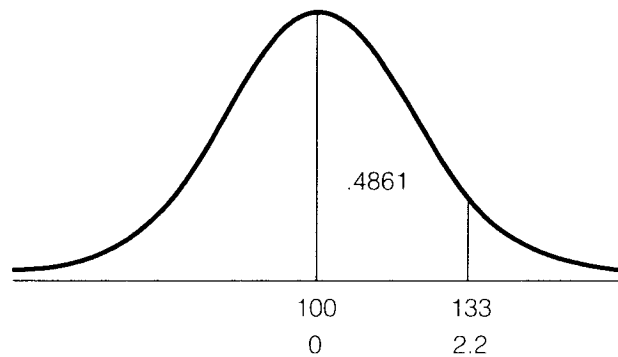
6.15   A standard IQ test produces normally distributed scores with a mean = 100 and a standard deviation of 15. A class of science students is grouped homogeneously by excluding students with IQ scores in either the top 5% or the bottom 5%. Find the lowest and highest possible IQ scores of students remaining in the class.

6.16   The weights of 18–24-year-old women are normally distributed with a mean of 132 pounds and a standard deviation of 27.4. If one randomly selected 150 of these women age 18–24 years, how many of them would be expected to weigh 100 to 150 pounds?

6.17   If blackout thresholds are normally distributed with a mean 4.7G and standard deviation of 0.8G, find the probability of randomly selecting a pilot with a blackout threshold that is less than 3.5G.

6.18   Your last statistics quiz had a mean of 30 and a standard deviation of 6. Assume a normal distribution.
a. What is the median?
b. What is the Z-score of the mean?
c. In order to get an A, your Z-score must be +1.5 or above. What is the minimum raw score necessary?
d. A Z-score of −2.0 and below will be an F. What is that raw score?
e. If your raw score is 27, what is your Z-score?
f. What raw score would be at the 95th percentile?
   Your kind and understanding statistics instructor decides to give everyone in the class an extra point on his or her raw score.
g. What is the new mean?
h. If you had a Z-score of −1.00 before the extra point, what is your Z-score after the extra point?
i. In order to get an A, you must still have a Z-score +1.5 from the mean. What is the minimum raw score necessary?
j. If your biostatistics instructor bases his grades on the normal curve (i.e., curves his grades), what effect will the extra point have on your grade?

6.19   Two hundred students took a test. The scores were normally distributed. Your score was in the 60th percentile. How many people scored at or below your score?

6.20   Serum cholesterol levels were taken from a population of college students. The results were normally distributed. Males had a mean of 195 and a standard deviation of 10. Females had a mean of 185 and a standard deviation of 12.
a. What were the cholesterol levels of the highest 5% of the males?
b. What were the cholesterol levels of the highest 5% of the females?
c. What percentage of males would have a cholesterol level of less than 180?
d. What percentage of females would have a cholesterol level of less than 180?
e. What percentage of males would have a cholesterol level between 180 and 200?
f. What percentage of females would have a cholesterol level between 180 and 200?

g. Concern was expressed by the health educators on a particular college campus of 10,000 that students with serum cholesterol levels above 200 might be at an increased risk of heart disease. If the campus was equally divided between males and females, how many males and how many females on this campus would be at an increased risk?

7 Sampling Distribution of Means

Chapter Outline

Learning Objectives

After studying this chapter, you should be able to

1. Distinguish between the distribution of a population and the distribution of its sample means

2. Explain the importance of the central limit theorem

3. Identify the main parts of the central limit theorem

4. Apply the principles of sampling distributions to predict the behavior of sample means

5. Compute and interpret the standard error of the mean

6. Determine when to use a *t* distribution

THE DISTRIBUTION OF A POPULATION AND THE DISTRIBUTION OF ITS SAMPLE MEANS

Statisticians are interested in drawing inferences about a population. For example, it would be prohibitively expensive to conduct a health status survey by giving everyone in the United States a standardized comprehensive physical examination. Instead, a statistician would recommend that a sample be examined to estimate the important health parameters of the population. Such estimates, being random variables, would be expected to vary from sample to sample. In fact, if we were to select a large number of samples from a population and tabulate the sample means, the result would be a distribution of sample means. And we might be surprised at the shape of that distribution.

It is of fundamental importance to make a clear distinction between a distribution of sample means and the **population distribution** of observations. A **distribution of sample means** is the set of values of sample means obtained from all possible samples of the same size (n) from a given population; that is, it is the population of all values of that statistic (sample mean in this case).

A distribution of sample means can be readily illustrated by again using the data of blood glucose measurements from the Honolulu Heart Study (Table 7.1 and Figures 7.1 and 7.2). Figure 7.1 illustrates the distribution of blood glucose values for the entire population of 7683 men. The population mean μ is 161.52,
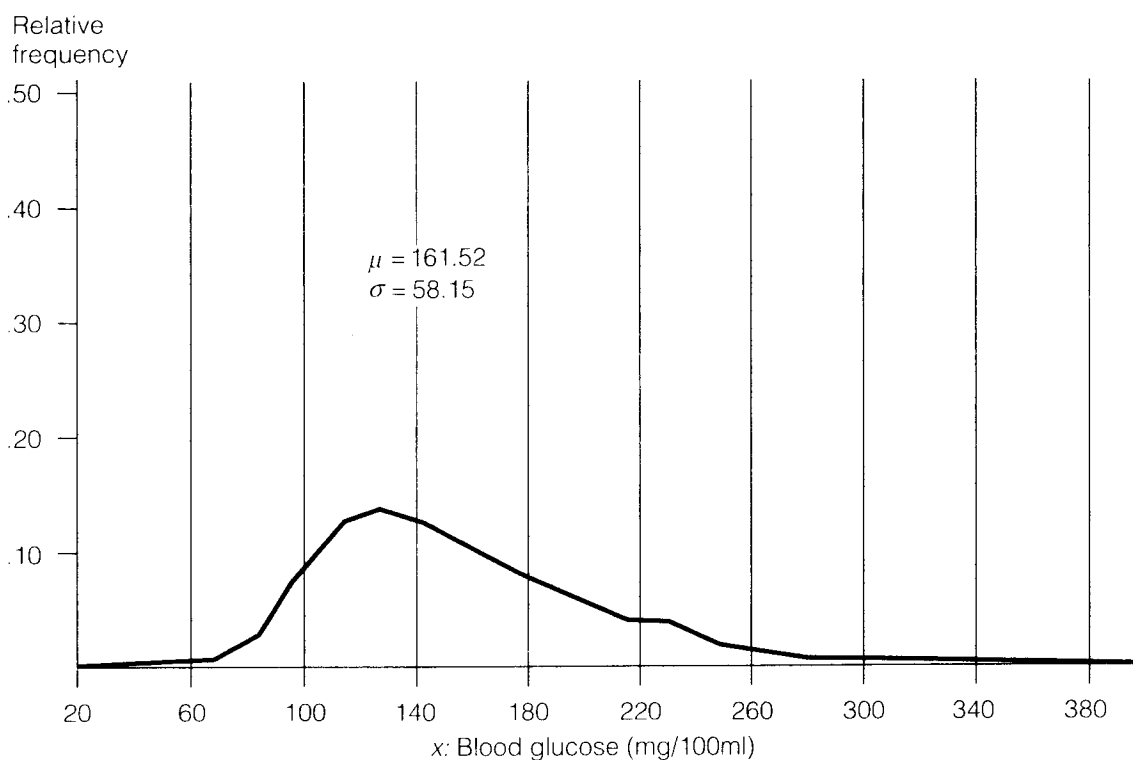
Figure 7.1   Distribution of Blood Glucose Values from the Honolulu Heart Study Population (N = 7683)

Table 7.1 Distribution of the Population and Distribution of Means from Samples for Blood Glucose Measurements of Men in the Honolulu Heart Study

Blood Glucose (mg/100 ml)	Number of Observations (frequency)	Sample Means ($n = 25$) (frequency)
30.1– 45.0	2	
45.1– 60.0	15	
60.1– 75.0	40	
75.1– 90.0	210	
90.1–105.0	497	
105.1–120.0	977	
120.1–135.0	1073	5
135.1–150.0	1083	62
150.1–165.0	849	201
165.1–180.0	691	109
180.1–195.0	569	23
195.1–210.0	440	
210.1–225.0	343	
225.1–240.0	291	
240.1–255.0	153	
255.1–270.0	115	
270.1–285.0	82	
285.1–300.0	60	
300.1–315.0	38	
315.1–330.0	18	
330.1–345.0	26	
345.1–360.0	19	
360.1–375.0	20	
375.1–390.0	9	
390.1–405.0	13	
405.1–420.0	11	
420.1–435.0	6	
435.1–450.0	5	
450.1–465.0	4	
465.1–480.0	24	
Total	7683	400

and its standard deviation σ is 58.15. These parameters are based on all 7683 cases. Suppose you select a sample of size 25 from this population and compute its sample mean $\bar{x}$ and standard deviation s. If, with $n = 25$, you repeat this random sampling scheme a number of times, you will generate a new distribution, that of the means of the samples. This particular random sampling was done 400 times to generate the distribution of sample means, as shown in the right-hand column of Table 7.1. If it were possible to select all possible samples of size 25 from the population of 7683, the result would be 8.524×10^{71} samples, an overwhelmingly large number! (In practice, of course, we take only one sample.)

Relative
frequency

.50

.40

.30 $\mu_{\bar{x}} = 160.66$
             $\sigma_{\bar{x}} = 12.24$
             (standard deviation of sample means)

.20

.10

   20   60   100   140   180   220   260   300   340   380
             $\bar{x}$: Mean blood glucose (mg/100ml)
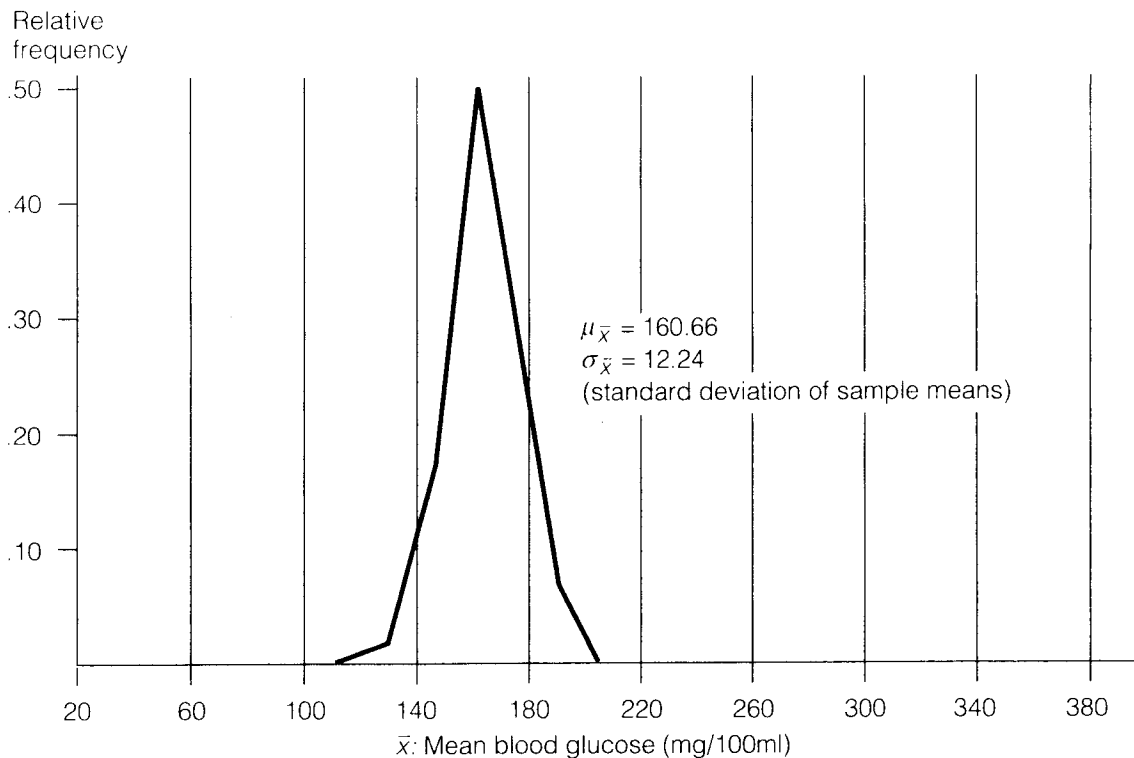
Figure 7.2   Distribution of Means of Samples of Blood Glucose ($n = 25$) from the Honolulu
Heart Study

As you can see in Figure 7.2, the distribution of sample means is symmetri-
cal, roughly bell-shaped, and centered close to the population mean of 161.52,
but with considerably less variation than the distribution of individual glucose
values.

CENTRAL LIMIT THEOREM

A quick glance at Figures 7.1 and 7.2 shows one striking similarity and an
equally striking difference. The mean of the distribution of sample means is al-
most identical to the mean of the underlying population. On the other hand, the
variability of sample means is far less than that of the population. This differ-
ence is quite evident from the broad, flat curve of blood glucose readings as
compared to the narrow, peaked curve of their means. Another noteworthy
characteristic is that the distribution of sample means is approximately bell-
shaped and symmetrical, whereas the original population distribution was no-
ticeably skewed. This may appear to be unusual, even paradoxical. Indeed it is!
It is one of the most remarkable features of mathematical statistics, and is called
the *central limit theorem.*

The **central limit theorem** states that for a randomly selected sample of size n (n should be at least 25, but the larger n is, the better the approximation) with a mean μ and a standard deviation σ:

1. The distribution of sample means x is approximately normal regardless of whether the population distribution is normal or not.

   From statistical theory come these two additional principles:

2. The mean of the distribution of sample means is equal to the mean of the population distribution—that is, $\mu_x = \mu$.

3. The standard deviation of the distribution of sample means is equal to the standard deviation of the population divided by the square root of the sample size—that is,

$$\sigma_x = \frac{\sigma}{\sqrt{n}} \tag{7.1}$$

We illustrate these three principles in Figure 7.3, which shows four very different population distributions. For each, as the sample size n increases, the sampling distribution of the mean approaches normality, regardless of whether the original population distribution was normal. A close scrutiny also reveals that, for any population distribution, the mean of each sampling distribution is the same as the mean (μ) of the population itself. Note also that as the sample size increases, the variability of the sampling distribution becomes progressively smaller.

3   STANDARD ERROR OF THE MEAN

The measure of variation of the distribution of sample means, $\sigma/\sqrt{n}$, referred to as the **standard error of the mean,** is denoted as $SE(x)$—that is,

$$SE(x) = \sigma_x = \frac{\sigma}{\sqrt{n}} \tag{7.2}$$

$SE(\bar{x})$ is a counterpart of the standard deviation in that it is a measure of variation, but variation of sample means rather than of individual observations. It is an important statistical tool because it is a measure of the amount of sampling error. Sampling error differs from other errors in that it can be reduced at will, provided one is willing to increase the sample size. A nearly universal application of the standard error of the mean in medical literature is to specify an interval of $x \pm 2SE(\bar{x})$, which includes the population mean, μ, with about 95% probability.

To prove the central limit theorem requires a considerable mathematical background beyond the level of this book. However, the sampling experiment
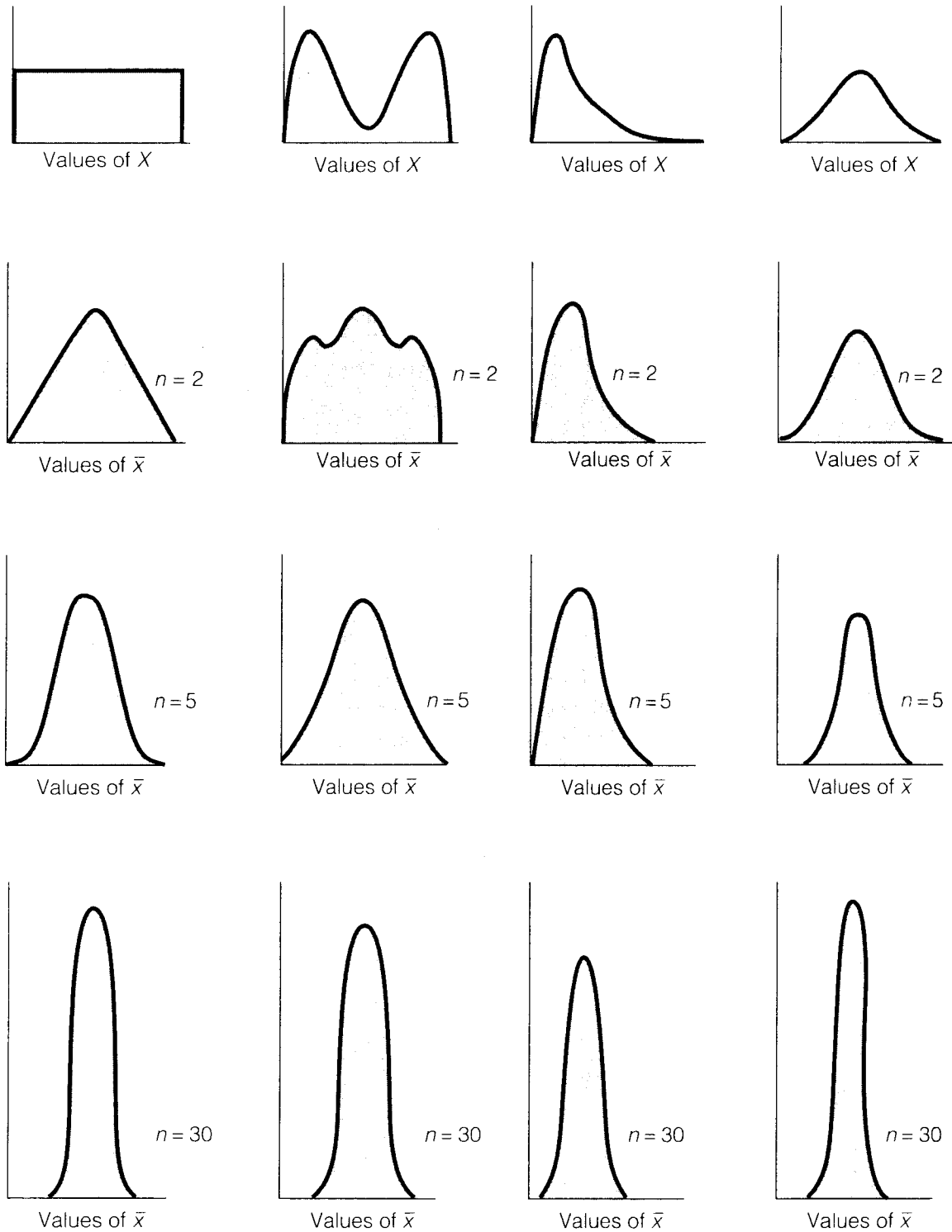
Figure 7.3 The Effect of Shape of Population Distribution and Sample Size on the Distribution of Means of Random Samples

of Figures 7.1 and 7.2 is in itself convincing evidence of the theorem's truthful-
ness. In these figures we can see the following:

1. The mean of the distribution of sample means μ_x is identical to the popu-
   lation mean μ.

2. The standard deviation of the sample means computed by use of the tra-
   ditional formula $\sqrt{\Sigma(x - \mu_x)^2/(n - 1)}$ is 12.24, very close to the standard
   error of the mean computed by using $\sigma_x = \sigma/\sqrt{n} = 11.63$. This is an im-
   pressive result; it is now possible to compute the standard error of the
   mean knowing only the sample size and the population σ or its estimate s.

3. The distribution of sample means is approximately normally distributed.

In practice σ is seldom known. We estimate it from the sample standard de-
viation s; consequently, the equation most commonly used for computing the
standard error of the mean is

$$s_x = \frac{s}{\sqrt{n}} \tag{7.3}$$

Note that $s_{\bar{x}}$ is estimated from a sample when σ is unknown.

Often we encounter data that are not normally distributed. This situation
may present a problem in statistical analysis; but by working with sample
means, we can meet the assumption of normality, providing the sample size is
sufficient (about 25 or more).

Because the central limit theorem states that sample means are approxi-
mately normally distributed, it is possible to find the area under the curve for
the normal distribution of sample means. To find it, we must again use the Z
transformation—that is, compute a Z score. For sample means, the equation for
Z is

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \tag{7.4}$$

This computed Z also establishes the relative position of x in a distribution of
sample means.

7.4   STUDENT'S t DISTRIBUTION

All too often the population standard deviation σ is unknown. Without σ we are
unable to calculate the Z score. We know, however, that when σ is unknown, it
may be estimated by s, the sample standard deviation. In Chapter 3, we calcu-
lated s like this:

$$s = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n - 1}}$$

Can this *s* be used instead of the σ in equation 7.4? Fortunately, yes. But we no longer have the standard normal distribution. Instead we have a distribution that was discovered in 1906 and published in 1908 by William S. Gossett, an English chemist and statistician employed by the Guinness Brewery in Dublin. Because the brewery, fearing release of trade secrets, rarely permitted publications by its employees, Gossett published under the pseudonym "Student." So his distribution is commonly referred to as **Student's *t* distribution.** The equation for its *t* score is

$$t = \frac{x - \mu}{s/\sqrt{n}} \tag{7.5}$$

This *t* distribution is similar to the standard normal distribution in that it is unimodal, bell-shaped, and symmetrical, and extends infinitely in either direction. Further, although the curve has more variance than the normal distribution, its area still equals 1.0. Areas under the curve, designated as α in Table B (inside back cover), are a function of a quantity called **degrees of freedom** (df), where

$$df = n - 1 \tag{7.6}$$

when estimating the standard deviation from a single sample. Degrees of freedom measure the quantity of information available in one's data that can be used in estimating the population variance σ^2. Therefore, they are an indication of the reliability of *s*, in that the larger the sample size, the more reliable *s* will be as an estimate of σ. It follows that the variance of the *t* distribution of means from large samples is less than those from small samples. Note that when the sample size exceeds about 30, the *t* distribution so closely approximates the normal distribution that for practical purposes the normal distribution may be used. In other words, for large samples, *s* becomes a quite reliable estimate of σ, as graphically illustrated in Figure 7.4. From the figure we can see that there are many *t* distributions, one for each degree of freedom.

The *t* distribution introduces the concept of infinite degrees of freedom for large sample sizes. In fact, the *t* distribution for infinite degrees of freedom is precisely equal to the normal distribution. This equality is readily seen by comparing the critical values for df $= \infty$ (infinity) of Table B for various values of α with those of Table A. The approximation is good, beginning with 25 df and nearly identical at 30 df. The percentage points of the *t* distribution in Table B are given for a limited number of areas. For example, the *t* value for $\alpha = .05$ with 15 df equals 1.753. It is found by locating df = 15 in the margin and
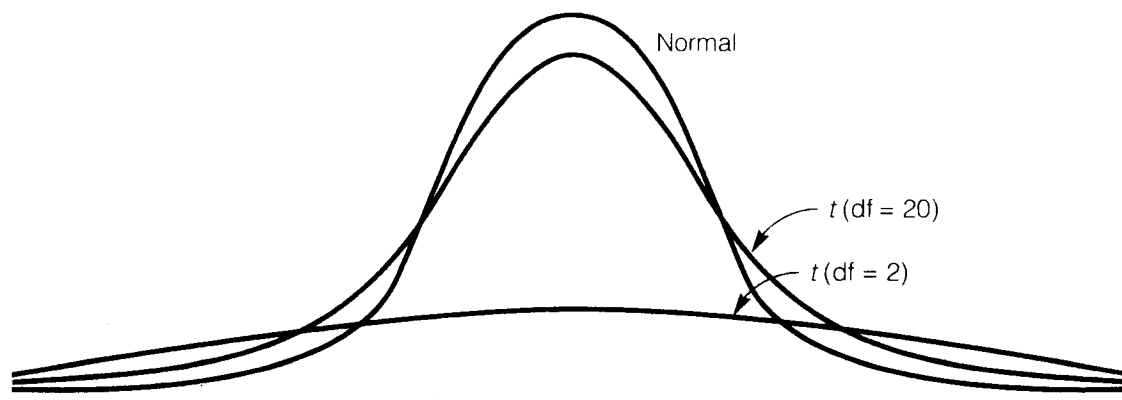
Figure 7.4   Comparison of *t* Distributions and Normal Distributions

reading the value of $t = 1.753$ in the column labeled $\alpha = .05$. Here α denotes the area in the tail under the curve.

When should the *t* distribution be used? Use it when the population standard deviation is not known. If you know the population's σ, or your sample exceeds 25, feel confident to use the normal distribution. Otherwise, the *t* distribution is indicated.

To summarize, Table 7.2 presents the equations for Student's *t* distribution, along with other equations introduced in this chapter.

Table 7.2 Characteristics of a Population Distribution and Its Distribution of Sample Means

Characteristic	Population Distribution	Distribution of Sample Means
Mean	μ	$\mu_x = \mu$
Measure of variation	σ	$\sigma_x = \dfrac{\sigma}{\sqrt{n}}$
Z score	$Z = \dfrac{x - \mu}{\sigma}$	$Z = \dfrac{x - \mu}{\sigma/\sqrt{n}}$
t statistic		$t = \dfrac{x - \mu}{s/\sqrt{n}}$

7.5   APPLICATION

Using the blood glucose observations from the entire Honolulu Heart Study population (Figure 7.1), we find that $\mu = 161.52$ and $\sigma = 58.15$. Suppose we select samples of size 25 from this population. (1) What proportion of sample

means would have values of 170 or greater? (2) What proportion of sample means would have values of 155 or lower?

For question 1, we reduce the problem to Z scores so we can determine the proportion of the area that is beyond Z. On obtaining

$$Z = \frac{170 - 161.52}{58.15/\sqrt{25}} = \frac{8.49}{11.63} = .73$$

we turn to Table A, which shows that the area to the right of $Z = .73$ is $.5 - .2673$, or about 23%.

For question 2, using the same technique, we can find the value of the relative deviate corresponding to the sample mean 155:

$$Z = \frac{155 - 161.25}{58.15/\sqrt{25}} = \frac{-6.25}{11.63} = -.54$$

Table A reveals that the area below $Z = -.56$ is $.5 - .2123 = .2877$, or about 29%.

ASSUMPTIONS NECESSARY TO PERFORM *t* TESTS

To perform a test of hypotheses the following two assumptions need to be met:

a. That the observations are randomly selected
b. That the distribution is a normal distribution

Sometimes the assumptions are not met, and individuals performing the *t* test still obtain valid results because the *t* test has a characteristic referred to as *being robust*. In other words, it can handle the violation of the assumptions.

Conclusion

A distinction exists between the distribution of a population's observations and the distribution of its sample means. A powerful tool called the central limit theorem gives reassuring results: No matter how unlike normal a population distribution may be, the distribution of its sample means will be approximately normal, provided only that the sample size is reasonably large ($n \geq 30$). The mean of the sampling distribution is equal to the mean of the population distribution. The standard error of sample means equals the standard deviation of the observations divided by the square root of the sample size. In sampling experiments, these results are often applied to determine how unusual a sample mean is.

Vocabulary List

central limit theorem	distribution of sample	standard error of the
degrees of freedom	means	mean
	population distribution	Student's t distribution

Exercises

7.1 Suppose samples of size 36 are drawn from the population of Exercise 6.5. Describe the distribution of the means of these samples.

7.2 If samples of size 25 are selected from the population of Exercise 6.6, what percentage of the sample means would you expect to be
a. between 57 and 63?
b. less than 58?
c. 61 or larger?

7.3 Repeat Exercise 7.2, but this time use a sample size of 64.

7.4 After completing Exercises 7.2 and 7.3, explain the effect of an increasingly larger sample size on the probabilities you calculated in Exercises 7.2 and 7.3.

7.5 Refer to the population of Exercise 6.9.
a. What is the standard error of the mean for $n = 16$?
b. What is the standard error of the mean for $n = 64$?
c. What is true about the relationship between n and $SE(\bar{x})$?

7.6 Suppose heights of 20-year-old men are approximately normally distributed with a mean of 71 in. and a population standard deviation of 5 in. A random sample of 15 20-year-old men is selected and measured. Find the probability that the sample mean $\bar{x}$
a. is at least 77 in.
b. lies between 65 and 75 in.
c. is not more than 63 in.

7.7 If the length of normal infants is 52.5 cm and the standard deviation is 4.5 cm, what is the probability that the mean of a sample of (a) size 10 and (b) size 15 is greater than 56 cm?

7.8 Suppose that the mean weight of infants born in a community is $\mu = 3360$ g and $\sigma = 490$ g.
a. Find $P(2300 < x < 4300)$.
b. Find $P(x \leq 2500)$.
c. Find $P(x \geq 5000)$.
What must you assume about the distribution of birthweights to make the answers to (a), (b), and (c) valid?

7.9 Suppose you select a sample of 49 infants from the population described in Exercise 7.8.
a. What are the mean and standard error of this sampling distribution?
b. Find $P(3100 < x < 3600)$.
c. Find $P(\bar{x} < 2500)$.
d. Find $P(\bar{x} > 3540)$.

What must you assume about the distribution of birthweights to make the answers to (b), (c), and (d) valid?

7.10 If the mean number of cigarettes smoked by pregnant women is 16 and the standard deviation is 8, find the probability that in a random sample of 100 pregnant women the mean number of cigarettes smoked will be greater than 24.

7.11 a. Describe the three main points of the central limit theorem.
b. What conditions must be met for the central limit theorem to be applicable?
c. Explain why the central limit theorem plays such an important role in inferential statistics.

7.12 a. Describe the difference between the distribution of observations from a population and a distribution of its sample means.
b. What are the differences between the standard deviation and the standard error?
c. When would we want to use the standard deviation and when the standard error?

7.13 a. Describe the difference between the Z and the t distributions.
b. Under what condition is the t distribution equivalent to the Z distribution?
c. If you had the choice of using the Z distribution or the t distribution, which would you use? Why?

7.14 If the cholesterol level of men in the community is normally distributed with a mean of 220 and a standard deviation of 50, what is the probability that a randomly selected sample of 49 men will have a mean between 200 and 240?

7.15 Compare the critical value ($Z = \pm1.96$) that corresponds to 5% of the tail area of the normal distribution with the critical values of the t distribution for df = 9, 19, 29, and ∞. As the degrees of freedom increases (which means that the sample size increases) what happens to the value of t compared with the value of Z? Explain why this is occurring.

7.16 If the forced vital capacity of 11-year-old white juvenile males is normally distributed with a mean of 2400 cc and $\sigma = 400$, find the probability that a sample of $n = 64$ will provide a mean
a. greater than 2500
b. between 2300 and 2500
c. less than 2350

7.17 a. Find the standard error in Exercise 7.16.
b. If you want the SE($\bar{x}$) to be one-half its size, how large a sample would you need to have?

7.18 Suppose systolic blood pressure of 17-year-old juvenile females is approximately normally distributed with a mean of 128 mmHg and a standard deviation of 12 mmHg.
a. What proportion of girls would you expect to have blood pressures between 122 mmHg and 134 mmHg?
b. If you were to select a sample of 16 girls and obtain their mean systolic blood pressure, what proportion of such samples would you expect between 122 mmHg and 134 mmHg?
c. Compare the results of (a) and (b) and explain the reason for the difference.

7.19   For data that are normally distributed, how much area is included under the normal curve
   a. within $\pm 1\sigma$?
   b. with ± 1 SE(x) for a distribution of sample means?
   Compare (a) and (b) and state why the results do or do not surprise you.

7.20   For Table 2.2, $X = 73$ and $\sigma^2 = 121$. If a person is chosen at random, what is the probability that she or he would have a diastolic blood pressure
   a. between 80 and 100?
   b. less than 70?
   c. greater than 90?

7.21   The mean blood glucose in Table 3.1 is 152 and $\sigma = 55$. Find the probability that a randomly selected individual would have a glucose value
   a. between 80 and 120
   b. less than 80
   c. greater than 200

7.22   If the mean serum cholesterol in Table 3.1 is 217 and the variance is 750, determine the probability that a randomly selected person would have a cholesterol value
   a. between 150 and 250
   b. greater than 250
   c. less than 150

7.23   For data that are normally distributed with a mean of 15 and $s = 40$, determine the proportion of individuals who would fall
   a. below 100
   b. between 100 and 200
   c. above 160
   d. below 160

7.24   If you selected a sample of $n = 100$ from the population given in Exercise 7.23, find the probability of obtaining an x below 160.

7.25   Find the SE(x) in Exercise 7.24.

7.26   Redo Exercise 7.8 but substitute $\sigma = 460$ g for the 490 g.

7.27   If you are sampling an obviously nonnormal population, what other fact can you use to permit you to justify performing tests of hypotheses?

7.28   If adult male cholesterol is normally distributed with mean $= 200$ and $\sigma = 35$, what is the probability of selecting a male whose value is 136?

7.29   A company that cans soup lists the number of milligrams of sodium as 950 mg per serving. A consumer group is concerned that the soup contains more sodium than is listed on the can. Do the cans contain more than the 950 mg of sodium per serving listed on the label? Assume a normal distribution.
   a. A sample of 25 cans has a mean sodium content of 975 mg per serving and a sample standard deviation of 60 mg. Write your null and research hypotheses.

b. How many degrees of freedom do you have?
c. What is your critical value? (Use a level of significance of .05.)
d. What is your calculated value?
e. State your conclusions. Be specific.
f. If you used a level of significance of .01 instead of .05, would your conclusion be any different? Explain.

8 Estimation of Population Means

Chapter Outline

8.1 Estimation
Explains why estimation is a primary statistical tool

8.2 Point Estimates and Confidence Intervals
Discusses point estimates and confidence intervals as two ways of estimating population parameters where only sample statistics are known

8.3 Two Independent Samples
Describes the difference between sample means as a modification of the estimate of a single-sample mean

8.4 Confidence Intervals for the Difference Between Two Means
Shows how confidence intervals help estimate the difference between two population parameters

8.5 Paired _t_ Test
Presents pros and cons of using a treatment group as its own control

8.6 Determination of Sample Size
Offers methods for determining in advance the sample size needed to design an efficient study

Learning Objectives

After studying this chapter, you should be able to

1. Compute a confidence interval from a set of data for
   a. a single population mean
   b. the difference between two population means
2. State three ways of narrowing the confidence interval
3. Determine the sample size required to estimate a variable at a given level of accuracy
4. Distinguish between a probability interval and a confidence interval
5. List the pros and cons of performing a before-and-after experiment

ESTIMATION

One of the principal objectives of research is comparison: How does one group differ from another? Specifically, we may encounter such questions as: What is the mean serum cholesterol level of a group of middle-aged men? How does it differ from that of women? From that of men of other ages? How does today's level differ from that of a decade ago? What is the mean number of children per family in the United States? What is the difference in the mean number of cavities between children who drink fluoridated water and those who drink non-fluoridated water? What is the difference in oxygen uptake between joggers and nonjoggers?

These are typical questions that can be handled by the primary tools of classical statistical inference—*estimation* and *hypothesis testing.* The unknown characteristic (parameter) of a population is usually estimated from a statistic computed from data of a sample. Ordinarily, we are interested in estimating the mean and the standard deviation of some characteristic of the population. The purpose of statistical inference is to reach conclusions from our data and to support our conclusions with probability statements. With such information, we will be able to decide if an observed effect is real or due to chance. Estimation is the main focus of this chapter. In the next chapter, we move to hypothesis testing.

In both estimation and hypothesis testing we may deal either with the characteristic of a population or with the differences in two population characteristics. Although the latter is more typical, the former is also quite commonly used. Either approach can be followed in one of two ways: (1) by estimating the difference in means between an experimental group and a control group or (2) by estimating the difference in means between one group before treatment and the same group after treatment.

In the first case, we deal with two random samples from two different populations; in the second, with two samples obtained from the same group before and after treatment. Also, in the first case, the observations are independent; in the second, the observations are not independent because they were obtained from the same source although at two different times. Because the estimation procedures are different for the two cases, they are treated separately.

POINT ESTIMATES AND CONFIDENCE INTERVALS

There are two ways of estimating a population parameter: a *point estimate* and a *confidence interval* estimate.

A **point estimate** of the population mean μ is the sample mean x computed from a random sample of the population. A frequently used point estimate for the population standard deviation σ is s, the sample standard deviation. For example, in attempting to assess the physical condition of joggers, an investigator

used the maximal volume oxygen (VO_2) uptake method. He found that the point estimate of VO_2 for joggers was $x = 47.5$ ml/kg. Because x is a statistic, the point estimate varies from sample to sample. In fact, if the investigator had repeated the experiment a number of times, he would have found a range of x's, any one of which would be a point estimate of the same population parameter. So a weakness in the point estimate idea is that it fails to make a probability statement as to how close the estimate is to the population parameter. This flaw is remedied by use of a **confidence interval** (CI), the interval of numbers in which we have a specified degree of assurance that the value of the parameter was captured. A confidence interval allows us to estimate the unknown parameter μ and give a margin of error indicating how good our estimate is. Using nothing more complicated than the Z score, it is possible to derive the equation for an interval that has a known probability of including the population mean μ. By using this method, you can be confident, say, that 95% of all sample means based on a given sample size will fall within ± 1.96 standard errors of the population mean. This outcome can be stated algebraically in terms of Z scores:

$$P\left(-1.96 \leq \frac{x - \mu}{\sigma/\sqrt{n}} \leq 1.96\right) = .95 \tag{8.1}$$

A few simple manipulations lead from equation 8.1 to equation 8.2, an important expression. First, multiply by $\sigma/\sqrt{n}$:

$$P\left(-1.96 \frac{\sigma}{\sqrt{n}} \leq x - \mu \leq 1.96 \frac{\sigma}{\sqrt{n}}\right) = .95$$

Next, change signs:

$$P\left(1.96 \frac{\sigma}{\sqrt{n}} \geq -x + \mu \geq -1.96 \frac{\sigma}{\sqrt{n}}\right) = .95$$

Finally, add x:

$$P\left(x + 1.96 \frac{\sigma}{\sqrt{n}} \geq \mu \geq x - 1.96 \frac{\sigma}{\sqrt{n}}\right) = .95$$

For convenience, reverse the inequality signs. The result is

$$P\left(x - 1.96 \frac{\sigma}{\sqrt{n}} \leq \mu \leq x + 1.96 \frac{\sigma}{\sqrt{n}}\right) = .95 \tag{8.2}$$

Using equation 8.2 on repeated sampling, you can expect (with a probability of .95) the true population mean μ to be captured by the interval $x - 1.96(\sigma/\sqrt{n})$

to $x + 1.96(\sigma/\sqrt{n})$. The interval is referred to as the *95% confidence interval* of the population mean and is usually denoted as

$$95\% \text{ CI of } \mu = \bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}} \tag{8.3}$$

This procedure can be used for other probabilities. For example, the 99% confidence interval for μ is given by

$$99\% \text{ CI of } \mu = \bar{x} \pm 2.576 \frac{\sigma}{\sqrt{n}} \tag{8.4}$$

These confidence interval equations are not used very often because they suffer from a drawback: σ is usually unknown. But we have already established that when σ is unknown, we can estimate it by s, the sample standard deviation. Let us say we use the *(1 − α) 100% confidence interval* for a population mean μ, which is an interval constructed from sample data such that, upon repeated sampling, it will have a probability $1 - \alpha$ of containing the population mean. As before, to construct the interval, we use a t value (with $n - 1$ df) instead of the Z value. By using a procedure parallel to the one employed for equations 8.3 and 8.4, we can obtain the confidence interval when only s (not σ) is known:

$$(1 - \alpha)100\% \text{ CI for } \mu = \bar{x} \pm t \frac{s}{\sqrt{n}} \tag{8.5}$$

where $t(s/\sqrt{n})$ is the margin of error for the CI and is a measure of sampling error.

■ EXAMPLE 1

If we wished to estimate the mean VO_2 uptake for a population of joggers from a sample of 25, we could use the 95% confidence interval for μ. We already know that $\bar{x} = 47.5$ ml/kg and $s = 4.8$ for a sample of 25. In Table B we find that the t value for 24 df for the central 95% of the t distribution is 2.064. The 95% confidence interval is thus

$$95\% \text{ CI of } \mu = \bar{x} \pm 2.064 \frac{s}{\sqrt{n}}$$

$$= 47.5 \pm 2.064 \frac{4.8}{\sqrt{25}}$$

$$= 47.5 \pm 1.98$$

$$= (45.5, 49.5)$$

The result: Upon many repetitions of this experiment, we would expect 95% of such intervals, $x - 2.064s/\sqrt{n}$ to $x + 2.064s/\sqrt{n}$, to capture the population mean μ. The values 45.5 and 49.5 are the lower and upper 95% **confidence limits**. The interval, 45.5 to 49.5 ml/kg, is the 95% confidence interval. It is important to note that the confidence interval varies but not the population mean μ. ■

The confidence interval provides a range that captures the true value of the population mean with 95% probability. However, there is still a 5% chance that the interval does not capture μ—there is a 2.5% chance that μ actually lies above $Z = 1.96$ (or below $Z = -1.96$). Therefore we use $Z_{.975} = 1.96$ and $Z_{.025} = -1.96$ in calculating the upper and lower confidence limits.

It is important to note an interesting distinction: these intervals are referred to as *confidence intervals*, not *probability intervals*. Before we actually obtain specific confidence limits based on a sample, the equation is properly referred to as a probability statement. But once the specific confidence limits are calculated, the **a posteriori probability** (i.e., the probability derived from observed facts) that the interval contains the mean μ or it does not. Therefore, with typical caution, statisticians refer to it as a 95% confidence interval because there is 95% confidence that in the long run the intervals constructed in such a way will indeed contain the population mean.

It would be incorrect to say in Example 1 that the probability is 95% that the true μ falls between 45.5 and 49.5 ml/kg. It either falls or does not fall between these two values. Once the interval is fixed, there is no randomness associated with it nor is there any probability.

The 95% confidence interval is used quite commonly, as is the 99% confidence interval. Other percentages may be used but are less frequently encountered in practice.

8.3   TWO INDEPENDENT SAMPLES

Suppose we wish to extend our example of comparing the physical condition of joggers and nonjoggers, again using the VO_2 uptake criterion. To obtain **two independent samples,** we first compute VO_2 uptake means for the two groups: joggers ($\bar{x}_1$) and nonjoggers ($\bar{x}_2$). The next logical step is to compute $\bar{x}_1 - \bar{x}_2$, the difference in mean VO_2 uptake for the two samples. As you might expect, $\bar{x}_1 - \bar{x}_2$ is an estimate of $\mu_1 - \mu_2$, the difference between means of the two underlying populations. Just as we computed confidence intervals for the mean, so we also compute them for the difference between two means.

From the central limit theorem, mathematical statisticians are able to demonstrate that $x_1 - x_2$ is normally distributed with a mean of $\mu_1 - \mu_2$ and a variance equal to $\sigma_1^2/n_1 + \sigma_2^2/n_2$. Its square root is the **standard error of the difference between two means** and is often denoted as

$$SE(x_1 - x_2) = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \tag{8.6}$$

This equation should not be too surprising because $\bar{x}_1$ and $\bar{x}_2$ are each normally distributed with respective variances of σ_1^2/n_1 and σ_2^2/n_2. But the variance of the difference is the *sum* of the two individual variances. This is certainly reasonable if we realize that the variation of $\bar{x}_1 - \bar{x}_2$ cannot help but be more than that which would be expected for either $\bar{x}_1$ or $\bar{x}_2$ separately.

Finally, the equation for the calculation of the Z score is

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}} \tag{8.7}$$

In many cases, we compare a given phenomenon in a treated and an untreated population. Because the cases and controls are being drawn from the same population, it is reasonable to assume that $\sigma_1^2 = \sigma_2^2$, thereby simplifying equation 8.7 to

$$Z = \frac{(x_1 - x_2) - (\mu_1 - \mu_2)}{\sigma\sqrt{1/n_1 + 1/n_2}} \tag{8.8}$$

As before, σ^2 is seldom known. So again we estimate it by a sample variance obtained from the data. This procedure again moves us from the normal to the t distribution. In such a case, we actually obtain two different estimates of σ^2—namely, s_1^2 and s_2^2. If it is safe to assume that these two are an estimate of a common variance, σ^2, we can pool the two sample variances and obtain the **pooled standard deviations,** s_p, a single improved estimate of σ^2 (improved because it is based on a larger sample). We get the **pooled sample variance** by taking a weighted average of s_1^2 and s_2^2:

$$s_p^2 = \frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2} \tag{8.9}$$

Equation 8.9 takes the sum of squares of the two separate samples and divides them by the sum of the degrees of freedom. This procedure for computing s_p^2 provides an unbiased estimate of σ^2.

After computing s_p^2, we can obtain s_p simply by extracting the square root. We need s_p to compute the t score:

$$t = \frac{(x_1 - x_2) - (\mu_1 - \mu_2)}{s_p\sqrt{1/n_1 + 1/n_2}} \tag{8.10}$$

with $n_1 + n_2 - 2$ df.

8.4   CONFIDENCE INTERVALS FOR THE DIFFERENCE BETWEEN TWO MEANS

After estimating the difference between two population means, we take the next logical step and establish a confidence interval around the difference. The point estimate of the difference was given by $x_1 - x_2$; the confidence interval equation may be derived from the probability statement:

$$P\left(-1.96 \le \frac{(x_1 - x_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \le 1.96\right) = .95 \tag{8.11}$$

This derivation, parallel to that of equation 8.2, yields the following equation for the 95% confidence interval:

$$95\% \text{ CI for } \mu_1 - \mu_2 = x_1 - x_2 \pm 1.96\left(\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right) \tag{8.12}$$

The general equation for the confidence interval with an unknown σ is

$$(1 - \alpha)100\% \text{ CI for } \mu_1 - \mu_2 = x_1 - x_2 \pm t\left(s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right) \tag{8.13}$$

which uses a t score, where t is the value corresponding to the $1 - \alpha$ proportion of the central area with $n_1 + n_2 - 2$ df.

These formulas (equations 8.5 and 8.13) will not give us correct results if the data were collected as a random sample. Because outliers will affect the value of x, the true level of confidence will likely be affected. Consequently, outliers should be removed before calculating a confidence interval.

■ EXAMPLE 2

In estimating physical condition by means of maximal VO_2 uptake, it is found that for a random sample of 25 joggers, $x_1 = 47.5$ ml/kg with $s_1 = 4.8$ and that for 26 nonjoggers, $x_2 = 37.5$ ml/kg with $s_2 = 5.1$. From these results, it is possible to compute a confidence interval. This computation will help us estimate the magnitude of the true difference, $\mu_1 - \mu_2$. The 99% confidence interval is calculated by first using Table B to obtain the t value. In Table B use a p of .99 from the two-sided row. The actual df in this example is 49. The nearest df in Table B is 50, which yields a t of 2.678.

To proceed with the computation of the confidence interval, we will need the value of s_p, which we obtain using equation 8.9.

$$s_p = \sqrt{\frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}}$$

$$= \sqrt{\frac{4.8^2(24) + 5.1^2(25)}{25 + 26 - 2}}$$

$$= \sqrt{\frac{1203.21}{49}} = \sqrt{24.65} = 4.96$$

It follows that

$$99\% \text{ CI for } (\mu_1 - \mu_2) = \bar{x}_1 - \bar{x}_2 \pm t_{.005}\left(s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\right)$$

$$= 47.5 - 37.5 \pm 2.678(4.96)\sqrt{\frac{1}{25} + \frac{1}{26}}$$

$$= 10.0 \pm 3.721$$

$$= (6.28, 13.72)$$

Hence, we have 99% confidence that the difference of the population mean for VO_2 uptake for joggers versus nonjoggers falls between 6.28 ml/kg and 13.72 ml/kg. So $\mu_1 - \mu_2$, which is estimated to be 10.0 ml/kg, is quite likely to be within this confidence interval. Because both of the confidence limits are positive, the interval does not include the value zero. This means that whatever the true difference is, joggers almost surely have a higher VO_2 uptake than nonjoggers. This will be consistent with the results of the t test. ∎

If more samples were obtained from the same populations as those in our example, we would find different means, different standard deviations, and consequently different confidence intervals. On average, we would expect that 99% of them would capture the true difference ($\mu_1 - \mu_2$) and 1% would not.

Figure 8.1 shows 50 confidence intervals for the differences in mean systolic blood pressure between smokers and nonsmokers, as given in Table 3.1. Here we know the true value of $\mu_1 - \mu_2$: $131.89 - 129.05 = 2.84$. So in this case we can determine how many of these confidence intervals actually include the known value of $\mu_1 - \mu_2 = 2.84$. We find that only the 24th one does not. One out of 50 is 2%—a bit higher than expected. However, in a longer series we would expect the result to be closer to 1%.

Narrow confidence intervals are of the greatest value in making estimates, because they allow us to estimate an unknown parameter with little room for error. This attribute moves us to consider all possible ways of narrowing confidence intervals. As seen from the confidence interval for the single population

$$\mu_1 - \mu_2 = 131.89 - 129.05$$
$$= 2.84$$
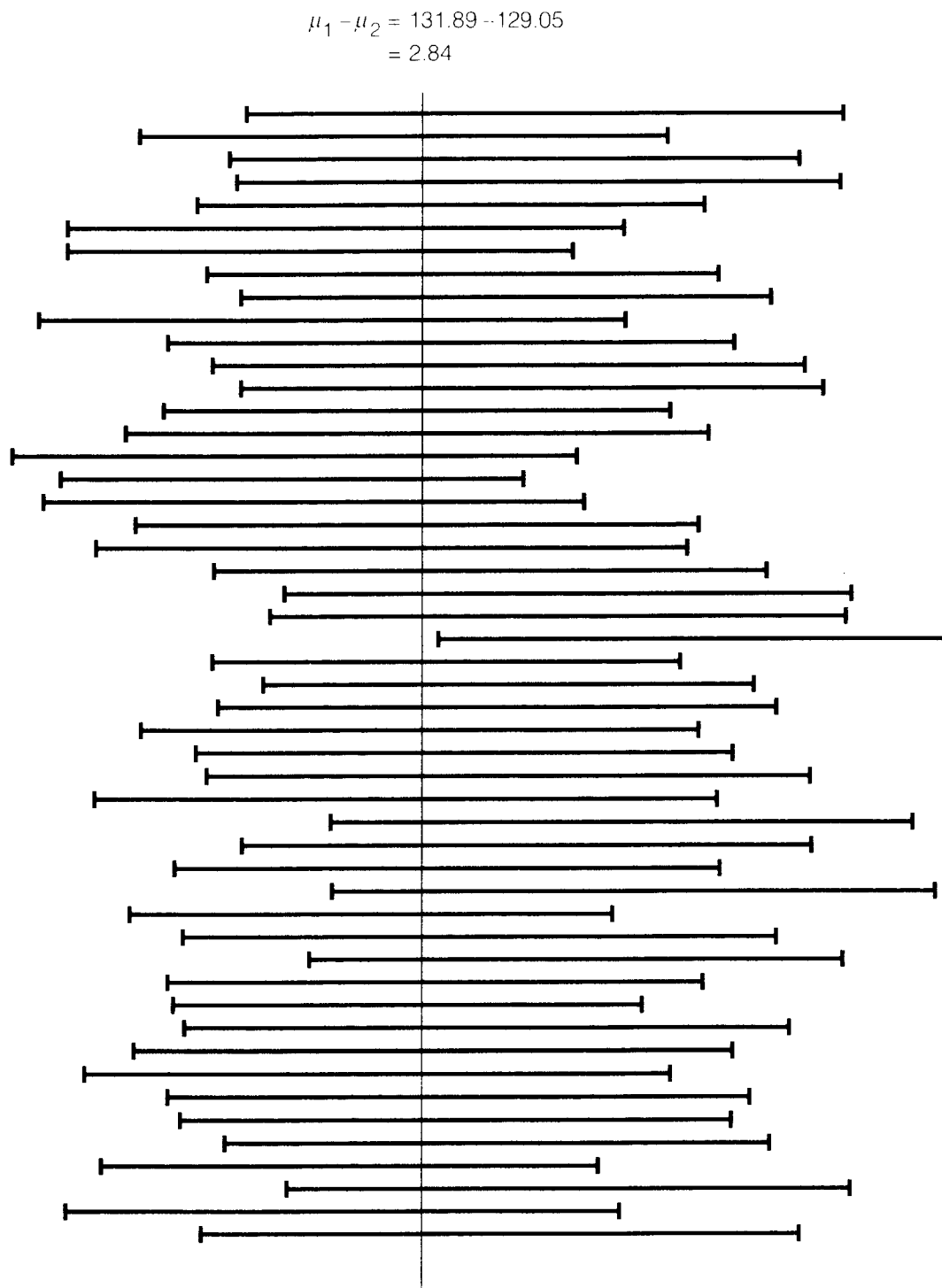
Figure 8.1 99% Confidence Intervals for Differences in Systolic Blood Pressure $\mu_1 - \mu_2$ for 50 Samples of Size 25 from Each Group of Nonsmokers and Smokers

mean, $x \pm Z(\sigma/\sqrt{n})$, the quantities that affect the width of the interval are the sample size, the Z value, and the standard deviation.

A confidence interval can be narrowed by

1. Increasing the sample size
2. Reducing the confidence level (for example, instead of using $Z = 2.58$ for 99% confidence, use $Z = 1.96$ for 95% confidence)
3. Increasing precision by reducing measurement (and other nonrandom) errors, thus producing a smaller variance

In Table 9.1 we present the confidence interval formula for the population mean μ and the difference of two populations $\mu_1 - \mu_2$. For the sake of convenience, we also include the confidence interval for π, the population proportion, and the confidence interval for $\pi_1 - \pi_2$, the difference of the population proportions we will discuss in Chapter 11.

5   PAIRED *t* TEST

In many investigations, the treatment group is used as its own control. This technique often generates quite appropriate comparisons because variability due to extraneous factors is reduced. It is not unusual for extraneous factors to account for many of the differences between means obtained from two independent samples. Given extraneous factors that add to variability, use of the treatment group as its own control will reduce the variability and give a smaller standard error, hence a narrower confidence interval. But we pay a price. First, independence is sacrificed in that we have two samples on the same items measured. Second, we are left with about half the degrees of freedom we would obtain using two independent samples. With fewer degrees of freedom, the t value is larger, and consequently the confidence interval is wider. So we must take these pros and cons into consideration when planning an experiment. Only then can we tell which procedure—two independent samples or a **paired t test**—will be more advantageous.

Data from paired t tests must never be thought of as coming from two independent samples. We can, however, handle the data statistically as a one-sample problem, then proceed with the confidence interval determination as for a single population mean. The procedure is to reduce the data to a one-sample problem by computing paired t tests for each subject. By doing this with paired observations, we get a list of differences that can be handled as a single-sample problem.

■ **EXAMPLE 3**

To determine whether a person's physical condition improves after taking up jogging, an investigator obtains maximal VO_2 uptake values before subjects start jogging and again six months later. Table 8.1 lists the values for VO_2 uptake for 25 randomly selected joggers. The difference between the before (x) and after (x') values is given as $d = x' - x$. The mean of the difference, d, is 12.42, and the standard deviation is $s_d = 1.57$. These values represent sample estimates of population parameters δ and σ_δ, respectively, where δ (delta) signifies the mean difference of population observations. We now can test whether jogging has been effective in improving physical condition as measured by the change in VO_2 uptake over time. Using the procedure for obtaining a single-sample confidence interval, we find that (with df $= n - 1 = 24$, where $n = $ number of pairs),

Table 8.1 Maximal Volume Oxygen Uptake Values of 25 Persons Age 30–40 Before and After They Became Joggers

Case	Before x	After x'	$d = x' - x$	d^2
1	34.1	47.9	13.8	190.44
2	32.3	44.6	12.3	151.29
3	36.5	47.3	10.8	116.64
4	38.6	50.6	12.0	144.00
5	39.6	51.9	12.3	151.29
6	31.8	43.3	11.5	132.25
7	31.0	43.3	12.3	151.29
8	38.8	51.9	13.1	171.61
9	29.3	41.2	11.9	141.61
10	35.3	47.6	12.3	151.29
11	41.3	54.0	12.7	161.29
12	43.3	55.6	12.3	151.29
13	33.8	45.6	11.8	139.24
14	28.3	39.4	11.1	123.21
15	36.8	48.9	12.1	146.41
16	30.6	42.4	11.8	139.24
17	28.8	46.3	17.5	306.25
18	40.0	52.8	12.8	163.84
19	39.8	48.9	9.1	82.81
20	44.8	56.7	11.9	141.61
21	30.8	46.5	15.7	246.49
22	25.8	38.7	12.9	166.41
23	32.7	44.2	11.5	132.25
24	35.3	47.2	11.9	141.61
25	37.9	51.0	13.1	171.61
	$\Sigma x = 877.3$	$\Sigma x' = 1187.8$	$\Sigma d = 310.5$	$\Sigma d^2 = 3915.27$
	$x = 35.1$	$x' = 47.5$	$d = 12.42$	

$$s_d = \sqrt{\frac{\Sigma d^2 - (\Sigma d)^2 / n}{n - 1}} = \sqrt{\frac{3915.27 - (310.5)^2 / 25}{24}} = 1.57$$

$$n = \text{\# of pairs}$$

$$99\% \text{ CI for } \delta = d \pm t_{.005} \frac{S_d}{\sqrt{n}}$$

$$= 12.42 \pm 2.797 \frac{1.57}{\sqrt{25}}$$

$$= 12.42 \pm .88$$

$$= (11.54, \ 13.30)$$

The sample estimate of δ, $d = 12.42$, indicates a gain in VO_2 uptake after jogging. The 99% confidence interval suggests that this gain is not likely to be less than 11.54 ml/kg or greater than 13.30 mg/kg. Note again that zero (i.e., the possibility that the before mean equals the after mean) is not included in the interval. The conclusion: six months of jogging improves one's physical fitness as measured by VO_2 uptake. ■

Paired t tests are one of several classes of experiments used with nonindependent samples. Other types include twin studies, studies of siblings of the same sex, litter mates in animal studies, and pairs of individuals who are matched on several characteristics such as age, race, sex, and condition of health. Because of the pairing, the test is known as a *paired t test.*

6   DETERMINATION OF SAMPLE SIZE

The daily life of a modern statistician involves a lot more than manipulating data and running computer programs. The statistician serves as a resource— sometimes to scientists, sometimes to administrators, almost always to persons less sophisticated in statistics. The statistician has to be prepared to answer many questions, and one of the most commonly heard is: "How large a sample size do I need to obtain a statistically meaningful result?"

Now that is a tough question. It is analogous, in a sense, to "How many runs must we score to win the ball game?" In the ball park, you could not field that one without more information, so you would have to ask a few questions yourself: "What's the score? What inning? Who's at bat? How many outs?" Similarly, in approaching the sample-size question, you first need to ask: "How much error can I live with in estimating the population mean? What level of confidence is needed in the estimate? How much variability exists in the observations?" Once you have the answers to these questions, you can attack the sample-size question.

Arithmetically, the sample size can be obtained by solving for n in the now-familiar equation

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \tag{8.14}$$

which could be rewritten as

$$Z = \frac{d}{\sigma/\sqrt{n}}$$

where $d = x - \mu$ and is a measure of how close we need to come to the population mean μ. Put another way, the estimate should be within d units of the population mean. Solving for n, we obtain

$$n = \left(\frac{Z\sigma}{d}\right)^2 \tag{8.15}$$

■ EXAMPLE 4

You need to estimate the mean serum cholesterol level of a population within 10 mg/dl of the true mean. You learn that $\sigma = 20$, and you want to state with 95% confidence that $\bar{x}$ is within 10 units of μ. So you obtain n as follows:

$$n = \frac{[(1.96)(20)]^2}{10^2} = 15.36$$

Because fractional sample sizes are not available, you conservatively round up to the next integer and get busy obtaining a sample of 16. If σ is unknown, you estimate it by s and use the t distribution. ■

Knowing how to determine sample size in advance of an experiment is wise planning, because your financial resources might limit you to, say, only 10 guinea pigs. If 16 are needed to gain significant results, it would be unwise to proceed. Alternatively, you could conserve resources by advance knowledge of the number of animals needed. If 16 guinea pigs would suffice, it would not be cost-effective to do the experiment with 30 animals.

Equation 8.15 is the simplest way of estimating sample size. In the next chapter, we look at other, somewhat more complicated approaches. Taken together, all these approaches underscore an important counsel to researchers: Consult a statistician to determine your sample size.

Conclusion

A point estimate is something of a "best guess" at a population parameter. A confidence interval gives us a range of values to which we can append a probability statement as to whether the population parameter is included. Differences between population means may be estimated in two ways: by use of two independent samples or by a single sample measured before and after the experiment. But first consider the pros and cons.

You should be prepared to answer the statistician's toughest and most commonly heard question: "How large a sample . . . ?" The answer is both easy and difficult—easy, in employing a simple equation; difficult, in getting the right input to that equation.

Vocabulary List

a posteriori probability	point estimate	standard error of the
confidence interval	pooled sample variance	difference
confidence limits	pooled standard	two independent
paired t test	deviation	samples

Exercises

8.1 Mice of a given strain were assigned randomly to two experimental groups. Each mouse was injected with a measured amount of tumor pulp. The pulp came from a large, suitable tumor excised from another mouse. After the tumor injections, the two groups received different chemotherapy treatments. Forty days after injection, the tumor volumes (in cubic centimeters) were measured as a comparison of the treatments. The data are as follows:

	Chemotherapy Treatment A	Chemotherapy Treatment B
n	27	30
x	.51 cc	.64 cc
s^2	.010	.045
	$s_p = .17$	

Estimate $\mu_1 - \mu_2$; calculate its 95% confidence interval.

8.2 Are the 95% confidence intervals narrower or wider than the 99% confidence intervals? Do Exercises 8.3 and 8.4; the results will either confirm your answer or cause you to change it.

8.3 The standard hemoglobin reading for healthy adult men is 15 g/100 ml with a standard deviation $\sigma = 2$ g. For a group of 25 men in a certain occupation, we find a mean hemoglobin of 16.0 g.
  a. Obtain a 95% confidence interval for μ and give its interpretation.
  b. Calculate the 95% confidence interval for the following sample sizes: 36, 49, and 64.
  c. As the sample size increases, do the confidence intervals shrink or widen? Explain. (*Hint:* Recall what you learned about the central limit theorem in Chapter 7.

8.4 Repeat Exercise 8.3 using a 99% confidence interval, instead of a 95% confidence interval.

8.5 The standard serum cholesterol for adult males is 200 mg/100 ml with a standard deviation of 16.67. For a sample of 49 overweight men the mean reading is 225.

a. Construct a 95% confidence interval for μ.

b. What size sample would you need to have 95% confidence that the estimate of μ is within 10 mg/100 ml?

8.6 The standard urine creatinine for healthy adult males is .25 to .40 g/6 hr.

a. If we assume the range encompasses 6 standard deviations, what is the estimate of the mean and the standard deviation of the population?

b. Construct the 99% confidence interval for μ.

8.7 The mean diastolic blood pressure of 100 individuals in Table 2.2 is 73 mmHg with a standard deviation $s = 11.6$ mmHg. Construct a 99% confidence interval for μ.

8.8 The mean weight of the sample of 100 men from the Honolulu Heart Study is 64 kg with the standard deviation $s = 8.61$. Obtain a point estimate and a 95% confidence interval for μ.

8.9 Compute 99% confidence intervals for $\mu_1 - \mu_2$ between males and females if, for 38 males, $x_1 = 74.9$ and $s_1^2 = 144$, and, for 45 females, $x_2 = 71.8$ and $s_2^2 = 121$.

8.10 Cholesterol measurements from 54 vegetarians and 51 nonvegetarians yield the following data:

Vegetarians:	115,	125,	125,	130,	130,	130,	130,	135,	135,	140,
	140,	140,	140,	145,	145,	150,	150,	150,	155,	160,
	160,	160,	160,	160,	165,	165,	165,	165,	165,	165,
	165,	170,	170,	170,	170,	170,	170,	170,	175,	175,
	175,	180,	180,	180,	180,	180,	185,	185,	185,	200,
	215,	215,	225,	230						
Nonvegetarians:	105,	110,	115,	125,	125,	130,	135,	145,	145,	150,
	150,	160,	165,	165,	165,	170,	170,	170,	170,	170,
	175,	175,	175,	180,	180,	180,	180,	185,	185,	190,
	190,	190,	190,	195,	200,	200,	200,	200,	200,	205,
	210,	210,	210,	210,	215,	220,	230,	230,	240,	240,
	245									

Find an estimate of $\mu_1 - \mu_2$ and calculate the 99% confidence interval for the difference between the population parameters.

8.11 a. Why is a confidence interval not called a probability interval?

b. What is the interpretation of a confidence interval?

c. What factors regulate the length of a confidence interval?

8.12 A hospital administrator wishes to estimate the mean number of days that infants spend in ICUs.

a. How many records should she examine to have 99% confidence that the estimate is not more than 0.5 day from the mean? Previous work suggests that $\sigma = 1.6$.

b. How many records should she examine if she wants to lower the confidence interval to 95%?

8.13 Find the 95% confidence interval for the difference in the mean systolic blood pressure of smokers and nonsmokers using the first 50 individuals of the Honolulu Heart Study population given in Table 3.1.

8.14 Obtain the 95% confidence interval for δ, the difference in cholesterol determinations obtained by two different labs on the same 10 patients as given in Exercise 9.15.

8.15 The weight gain for a control diet of $n_1 = 10$ individuals is $\bar{x}_1 = 12.78$ and for a treatment diet of $n_2 = 9$ individuals, $x_2 = 15.27$. The corresponding variances are $s_1^2 = 13.9$ and $s_2^2 = 12.8$. Compute the 90% confidence interval for $\mu_1 - \mu_2$.

8.16 The mean serum cholesterol level of 25 men ages 65–74 is 236, with $s_1 = 50$. For 25 women of the same age, the mean is 262, with $s_2 = 49$.
a. What is the 95% confidence interval for the difference in mean serum cholesterol level between men and women?
b. What is the 99% confidence interval?

8.17 The mean hemoglobin of $n_1 = 16$ white women is $\bar{x}_1 = 13.7$, with $s_1^2 = 2.3$, and for $n_2 = 20$ black women, $\bar{x}_2 = 12.5$, with $s_2^2 = 2.1$.
a. What is the 95% confidence interval for $\mu_w - \mu_b$, the difference between white and black women's hemoglobin?
b. What is the 99% confidence interval for $\mu_w - \mu_b$?

8.18 Repeat Exercise 8.15 using a 95% confidence interval.

8.19 Rework Exercise 8.16 after replacing s_1 with 60 and s_2 with 64.

8.20 Complete the following:
a. Perform a test of the cholesterol measures shown in Exercise 8.10 at the $\alpha = .01$ level.
b. Compare the result in (a) with the result of the 99% confidence interval and explain why there are or are not differences between the two.

8.21 Also obtain the 99% confidence interval for δ mentioned in Exercise 8.1.

8.22 Rework Exercise 8.15 after making the following changes: $n_1 = 25$, $x_1 = 15$, $n_2 = 16$, $x_2 = 16.9$, and $s_1^2 = 12$, and $s_2^2 = 10$. Compute the 95% confidence interval $(\mu_1 - \mu_2)$.

8.23 Using the data of Table 3.1, determine if there is a significant difference in the ponderol index between smokers and nonsmokers at the $\alpha = .05$ level.

9 Tests of Significance

Chapter Outline

9.1 Definitions
Before launching into a formal discussion of the technique, explains important concepts involved in a test of significance and presents an analogy

9.2 Basis for a Test of Significance
Illustrates the rationale for a test of significance by use of a specific example

9.3 Procedure for a Test of Significance
Gives a formal description of the steps that constitute a test of significance, and uses an example from the Honolulu Heart Study to illustrate the procedure

9.4 One-Tailed Versus Two-Tailed Tests
Explains how to decide whether a test of significance is to be unidirectional or bidirectional

9.5 Meaning of "Statistically Significant"
Emphasizes the fact that "significance" in a statistical sense differs from the ordinary meaning of the word and is related to the testing procedures

9.6 Type I and Type II Errors
Discusses the two types of errors one is liable to make in the performance of a test of significance

9.7 Test of Significance of Two Independent Sample Means
Uses an example of the difference in oxygen uptake of joggers and nonjoggers to illustrate the most common method of comparing sample means—the t test

9.8 Relationship of Tests of Significance to Confidence Intervals
Demonstrates how confidence intervals can be used to perform tests of significance

9.9 Summary Table of Inference Formulas

9.10 Sensitivity and Specificity
Gives formulas to calculate the percent of false positives and false negatives of a screening procedure

Learning Objectives

After studying this chapter, you should be able to

1. Outline and explain the procedure for a test of significance
2. Explain the meaning of a null hypothesis and its alternative
3. Define statistical significance
4. Find the value of Z or t corresponding to a specified significance level, α
5. Distinguish between a one-tailed and a two-tailed test
6. Distinguish between the critical value and the test statistic
7. Determine when to use a Z test and when to use a t test
8. Distinguish between the meaning of practical and technical significance
9. Determine whether the difference between two means is statistically significant for both independent and dependent sample means
10. Explain the meaning and relationship of the two types of errors made in testing hypotheses
11. Be able to list the reasons it is inappropriate to perform the test

$$ t = \frac{(x_1 - \bar{x}_2 - 0)}{s_p\sqrt{1/n_1 + 1/n_2}} $$

    on dependent sample means
12. Explain the meaning of a P value
13. Explain the relationship between a confidence interval and a test of significance and how the confidence interval can be used in testing a given hypothesis

9.1   DEFINITIONS

Before getting into the step-by-step procedure of a test of significance, you will find it helpful to look over the following definitions.

Hypothesis. A statement of belief used in the evaluation of population values.

Null hypothesis, H_0. A claim that there is no difference between the population mean μ and the hypothesized value μ_0.

Alternative hypothesis, H_1. A claim that disagrees with the null hypothesis. If the null hypothesis is rejected, we are left with no choice but to fail to reject the alternative hypothesis that μ is not equal to μ_0.

Test statistic. A statistic used to determine the relative position of the mean in the hypothesized probability distribution of sample means.

Critical region. The region on the far end of the distribution. If only one end of the distribution is involved, the region is referred to as a *one-tailed test*; if both ends are involved, the region is known as a *two-tailed test*. When the computed Z falls in the critical region, we reject the null hypothesis. The critical region is sometimes called the *rejection region*. The probability that a test statistic falls in the critical region is denoted by α.

Significance level. The level that corresponds to the area in the critical region. By choice, this area is usually small; the implication is that results falling in it do so infrequently. Consequently, such events are deemed unusual or statistically significant. When a test statistic falls in this area, the result is referred to as *significant* at the α level.

P value. The area in the tail or tails of a distribution beyond the value of the test statistic. The probability that the value of the calculated test statistic, or a more extreme one, occurred by chance alone is denoted by P.

Nonrejection region. The region of the sampling distribution not included in α; that is, it is located under the middle portion of the curve. Whenever a test statistic falls in this region, the evidence does not permit us to reject the null hypothesis. The implication is that results falling in this region are not unexpected. The nonrejection region is denoted by $(1 - \alpha)$. Some incorrectly refer to it as the "acceptance region." However, to call it such is misleading because it has only a probability of occurring in it.

Test of significance. A procedure used to establish the validity of a claim by determining whether or not the test statistic falls in the critical region. If it does, the results are referred to as significant. This test is sometimes called the *hypothesis test*.

To reinforce some of these definitions, let us consider an analogy. In a criminal court, the jury's duty is to evaluate the evidence of the prosecution and the defense to determine whether a defendent is guilty or innocent. By use of the judge's instructions, which provide guidelines for their reaching a decision, the members of the jury can arrive at one of two verdicts: guilty or not guilty. Their decision may be correct or they could make one of two possible errors: convict an innocent person or exonerate a guilty one.

A court trial and a **test of significance** have a lot in common. By a statistical test of significance, one attempts to determine whether a certain claim is valid. The claim is usually stated as a **null hypothesis,** H_0, which holds that the mean of a certain population is some value, μ_0 (the defendent is innocent). Using the data obtained in the sample (the evidence), one computes a **test statistic** (the jury) and uses it to determine whether it supports the null hypothesis claim (innocence) that the sample comes from a population with a mean of μ_0. The basis for finding out whether the test statistic supports the null hypothesis is the **critical region** (judge's instructions). The critical region sets guidelines for rejecting or failing to reject the null hypothesis. If the computed statistic falls in the criti-

cal region of the distribution curve, where it is unlikely to occur by chance, the claim is not supported (conviction). If the test statistic falls in the **nonrejection region,** where it is quite likely to occur by chance, the claim is not rejected (possible exoneration). A look ahead to Figure 9.6 might clarify the discussion.

.2   BASIS FOR A TEST OF SIGNIFICANCE

The purpose of a test of significance is to determine what evidence the data provide to reject a specific null hypothesis; that is, we determine whether or not the data provide evidence against the supposition made by the null hypothesis, which supposes that there is not an effect, in favor of the **alternative hypothesis,** which supposes that there is an effect.

To illustrate the basic concepts of a test of significance, let us again consider the Honolulu Heart Study. Suppose someone claims that the mean age of the population of 7683 individuals is 53.00 years. How can you verify (or reject) this claim? Start by drawing a sample of, say, 100 persons. Suppose the sample mean equals 54.85. Now the question is: What is the likelihood of finding a sample mean of 54.85 in a sample of 100 from a distribution whose true mean, μ, is 53? You can determine the answer by examining the relative position of $\bar{x}$ (54.85) on the scale of possible sample means. In Figure 9.1 you can see that 54.85 falls considerably above the hypothesized population mean of 53.

If the probability of such an occurrence is small as judged by the areas (in either direction) in the tails beyond this point, the occurrence is considered unusual or statistically significant. Why consider the areas in *both* directions? Remember that x could have fallen either above or below the mean μ. If x fell close to the center of the distribution, the probability of its occurring by chance would be fairly high. Events that have a high probability of occurrence are common and consequently *not significant*. The likelihood (probability) of the chance occurrence of a sample mean falling as far from the population mean can be obtained by performing a test of significance.

$$\sigma_{\bar{x}} = .55$$

$\bar{x}$ 51.15 $\mu = 53$ 54.85
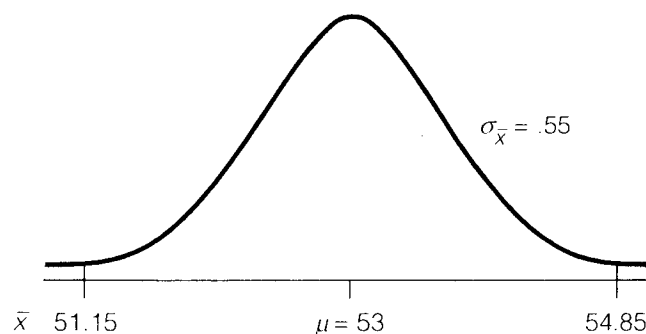
Figure 9.1   Distribution of Sample Means

9.3   PROCEDURE FOR A TEST OF SIGNIFICANCE

To perform a test of significance, we take the following steps:

1. State H_0: $\mu = \mu_0$ versus H_1: $\mu \neq \mu_0$.
2. Choose a significance level $\alpha = \alpha_0$ (usually $\alpha_0 = .05$ or $.01$).
3. Compute the test statistic (the Z score):

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$$

4. Determine the critical region, which is the region of the Z distribution with $\alpha/2$ in each tail, as shown in Figure 9.2.
5. Reject the null hypothesis if the test statistic Z falls in the critical region. Do not reject the null hypothesis if it falls in the nonrejection region.
6. State appropriate conclusions.
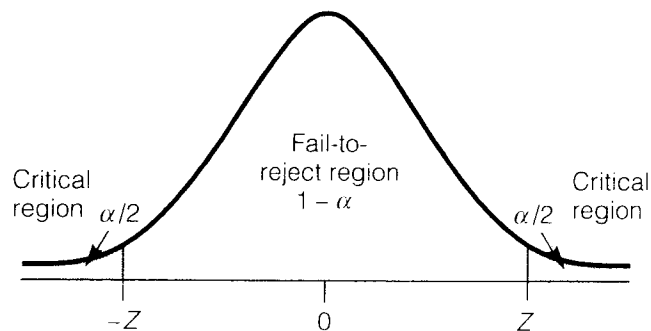
Figure 9.2   Critical Region of a Test Statistic

■ **EXAMPLE 1**

Using the Honolulu Heart Study sample of $n = 100$, which has a mean age $\bar{x} = 54.85$, we can perform a test of significance to determine the likelihood that such a sample mean comes from a population whose mean is 53, given that $\sigma = 5.50$. Using the procedure just outlined, we obtain the following:

1. H_0: $\mu = 53$ versus H_1: $\mu \neq 53$.
2. Significance level $\alpha = .05$.
3. Test statistic:

$$Z = \frac{x - \mu}{\sigma/\sqrt{n}} = \frac{54.85 - 53}{5.5/\sqrt{100}} = \frac{1.85}{.55} = 3.36$$

4. Critical region: From the Z distribution (Table A), we find, for a two-tailed test where $\alpha/2 = .025$, the corresponding $Z = \pm 1.96$ (Figure 9.3).

5. Because the computed test statistic $Z = 3.36$ (step 3) falls within the critical region (beyond the critical values ± 1.96), we are compelled to reject the null hypothesis that the sample comes from a population with a mean of 53 and not reject the alternative hypothesis that the sample comes from a population with a mean not equal to 53.
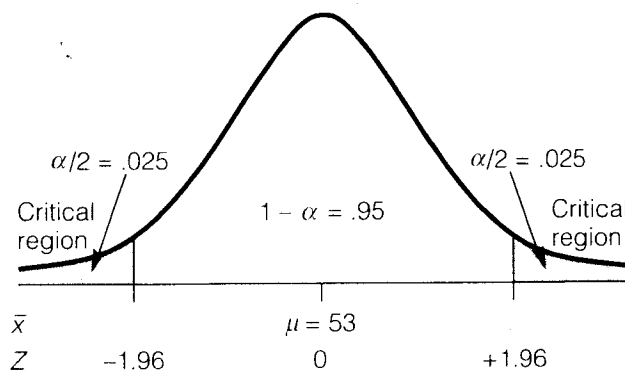
Figure 9.3 Critical Region for Example 1

This result is considered to be "significant at the $\alpha = .05$ level" because the probability of its occurring by chance is *less* than .05. The actual probability of obtaining a Z of 3.36 or larger is much smaller.

Because the computed test statistic falls 3.36 standard errors from the mean, we could say that the probability of having a sample mean of 54.85 or larger in either direction (that is, above or below $\mu = 53$) is less than .002. This figure is usually denoted by P and is obtained by summing the area beyond $Z = \pm 3.36$, which is at most $2(.5 - .4990) = 2(.001) = .002$. (Observe that because 3.36 does not appear in Table A, we use the area .4990 corresponding to 3.09, the largest value in the table.)

The P value of .002 indicates that the probability of selecting by chance a mean that falls as far as or farther than 3.36 standard errors above or below the population mean of 53 is quite small—that is, less than .002. You could ask yourself, "How could I be so lucky or unlucky as to have obtained such a result?" Your logical conclusion: The sample probably came from another population with a mean other than 53. ■

From Example 1, we can see that the test is based on how well x, the estimate of μ, estimates the parameter μ. If the H_0 is true, we would expect the $\bar{x} - \mu$ to be small. If the H_1 is true, we would expect the $\bar{x} - \mu$ to be large. By comparing the difference $\bar{x} - \mu$ relative to the SE($\bar{x}$)—that is, computing the test statistic— we can estimate the probability that this test statistic provides evidence against the supposition made by the H_0. By examining where the test statistic falls on

the sampling distribution of computed Z's or t's, we can obtain the probability that this outcome supports the H_0 or the H_1. This probability is measured by the **P value.** The smaller the P, the stronger the evidence that the H_0 is false, and the larger the P, the stronger the evidence that H_1 is false. Specifically, we decide that a result is statistically significant if the P value is smaller than the value of α chosen to define the critical region.

9.4   ONE-TAILED VERSUS TWO-TAILED TESTS

In testing statistical hypotheses, you must always ask a vital question: "Am I interested in the deviation of x from μ in one or both directions?" The answer is usually implicit in the way H_0 and H_1 are stated. If you are interested in determining whether the mean age is significantly *different* from a given μ, you would perform a **two-tailed test,** because the deviation, $\bar{x} - \mu$, could be either negative or positive.
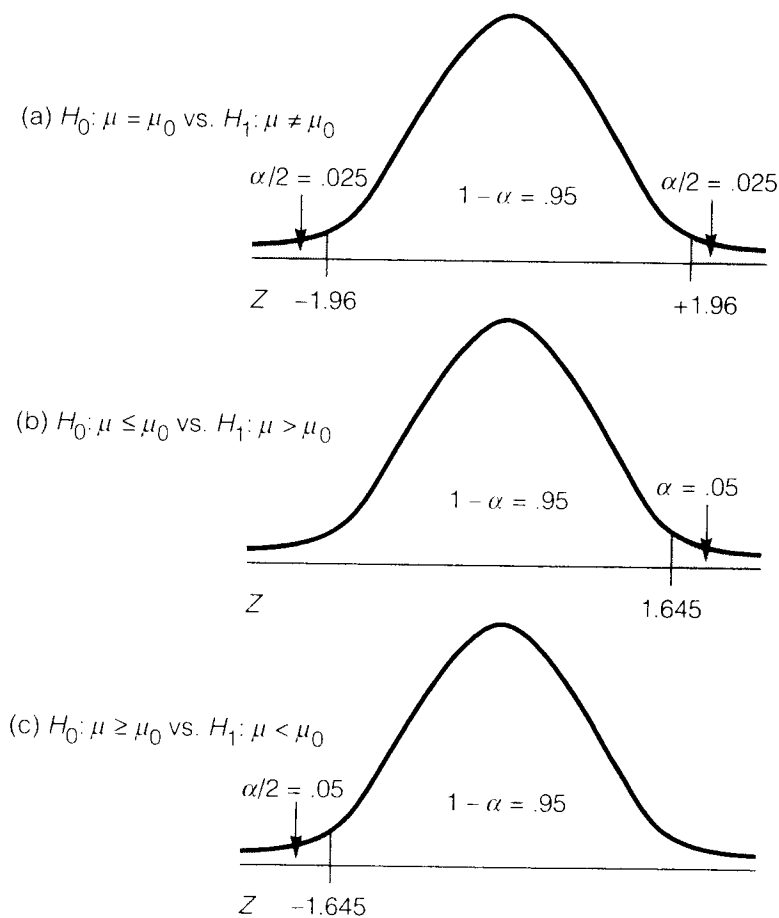
Figure 9.4   Two-Tailed Versus One-Tailed Test ($\alpha = .05$)

If you are interested in whether the mean age is significantly *larger* than the given μ, you would perform a **one-tailed test.** Likewise, you would go to the one-tailed test for mean ages *smaller* than μ.

Figure 9.4 illustrates the use of each kind of test. Figure 9.4a indicates that a two-tailed test is called for in testing the null hypothesis that $\mu = \mu_0$ against the alternative hypothesis that $\mu \neq \mu_0$. One half of the rejection region α is placed in each tail of the distribution; that is, we would reject H_0 if the value of the calculated test statistic fell in either of the outlying regions. Figure 9.4b indicates a one-tailed test for testing the null hypothesis that $\mu \leq \mu_0$ against the alternative hypothesis that $\mu > \mu_0$. Here, the critical region falls entirely in the positive tail; we would reject H_0 if the test statistic were so large as to fall in the critical region. Figure 9.4c indicates a left-handed one-tailed test for testing the null hypothesis that $\mu \geq \mu_0$. Here, the critical region falls entirely in the negative tail; we would reject H_0 if the calculated statistic were negative and fell in the critical region.

A one-tailed test is indicated for questions like these: Is a new drug superior to a standard drug? Does the air pollution level exceed safe limits? Has the death rate been reduced for those who quit smoking? A two-tailed test is indicated for questions like these: Is there a difference between the cholesterol levels of men and women? Does the mean age of a group of volunteers differ from that of the general population?

■ **EXAMPLE 2**

A smog alert is issued when the amount of a particular pollutant in the air is found to be greater than 7 ppm. Samples collected from 16 stations give an $\bar{x}$ of 7.84 with an s of 2.01. Do these findings indicate that the smog alert criterion has been exceeded, or can the results be explained by chance? Because σ is estimated by s, we rely on the t test.

1. H_0: $\mu \leq 7.0$ and H_1: $\mu > 7.0$.
2. $\alpha = .05$.
3. Test statistic:

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{7.84 - 7.0}{2.01/\sqrt{16}} = \frac{.84}{.50} = 1.68$$

4. Critical region: Because the H_1: $\mu > 7.0$ indicates a one-tailed test, we place all of $\alpha = .05$ on the positive side. From Table 7.2 we find that, for 15 df, $t_{.05} = 1.753$ (Figure 9.5).
5. Because the calculated $t = 1.68$ does not fall in the critical region, we do not reject H_0; alternatively, we conclude the data were insufficient to indicate that the critical air pollution level of 7 ppm was exceeded. ■
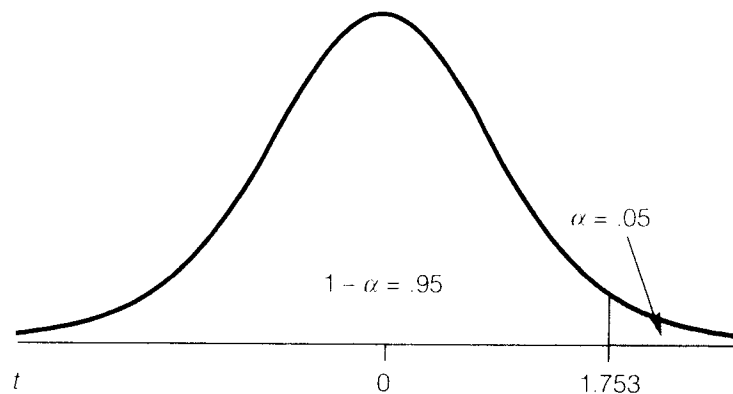
Figure 9.5   Critical Region for Example 2

9.5   MEANING OF "STATISTICALLY SIGNIFICANT"

Research reports often state that the results were **statistically significant,** ($P < .05$), or make some similar statement. Such a comment means that the observed difference is too large to be explained by chance alone. The **significance level,** somewhat arbitrarily selected at such values of α as .05, .025, .01, or .001, is a measure of how significant a result is. The significance level α is also the magnitude of error that one is willing to take in making the decision to reject the null hypothesis. Some investigators prefer to report their results in terms of the P value alone and let the reader conclude whether the information is sufficient to conclude that factors other than chance are operating.

In Section 9.3, a P value was calculated for the test of $\mu = 53$. Because it was a two-tailed test, we doubled the area in the tails beyond $Z = \pm 3.36$—namely, $P < .002$. For a one-tailed test, the P value would be the area beyond $Z = 3.36$— that is, $P < .001$. Researchers and statisticians generally agree on the following conventions for interpreting P values:

P value	Interpretation
$P > .05$	Result is not significant; usually indicated by no star.
$P < .05$	Result is significant; usually indicated by one star.
$P < .01$	Result is highly significant; usually indicated by two stars.

Some investigators would consider $P < .10$ to be marginally significant. "Statistically significant" means that the evidence obtained from the sample is not compatible with the null hypothesis; consequently, we reject H_0. However, just because a result is "not statistically significant" does not prove that H_0 is true. We may not be able to reject H_0 simply because the sample was too small to provide enough evidence to do so. In that sense, the decision to reject a null hypothesis is stronger than the decision not to reject it. Nor does "statistically

significant" imply clinically significant; that is, the difference, although techni-
cally "significant," may be so small that it has little biological or practical
consequence.

.6   TYPE I AND TYPE II ERRORS

In our analogy between hypothesis testing and a criminal trial, we noted that
the jury could make one of two errors: (1) reject the claim of innocence when the
defendent is indeed innocent or (2) fail to reject the claim of innocence when the
defendent is indeed guilty. Likewise, in testing a null hypothesis (H_0), you have
two possible decisions:

1. H_0 is false and consequently rejected; that is, the evidence is that the sam-
   ple comes from another population than one having $\mu = \mu_0$.
2. H_0 is true and consequently we fail to reject it. The observed difference
   between μ and μ_0 is relatively small and may be reasonably ascribed to
   chance variation.

If your decision is that H_0 is false when indeed it is, you have reached a cor-
rect decision. If you decide that H_0 is false when it is actually true, an event
likely to occur α proportion of the time, you have committed a **type I error** (also
referred to as an **α error**)—rejecting a true hypothesis that in the court analogy
corresponds to convicting an innocent person. If your decision is that H_0 is true
when indeed it is, you have also reached a correct decision. If you decide that H_0
is true when it is actually false, an event likely to occur β proportion of the time,
you have committed a **type II error** (also referred to as a **β error**)—accepting a
false hypothesis that in the court analogy corresponds to freeing a guilty per-
son. These two errors are summarized in Figure 9.6.

		TRUE STATE OF NATURE	
		H_0 is true	H_0 is false (H_1 is true)
D E C I S I O N	Accept H_0	Correct decision $(1-\alpha)$	Type II error (β)
	Reject H_0 (assume H_1 is true)	Type I error (α)	Correct decision $(1-\beta)$
	Total	1	1

$P(\text{Accept } H_0 | H_0 \text{ true}) = 1 - \alpha$

$P(\text{Reject } H_0 | H_0 \text{ false}) = 1 - \beta$

Figure 9.6   Possible Errors in Hypothesis Testing

In the test of a null hypothesis, some specific value for the parameter, say μ_0, is proposed. If this value happens to be correct but we reject it based on the observed sample, we have committed a type I error. If the proposed value happens to be incorrect but we accept it based on the observed sample, we have committed a type II error. Therefore, we can say that the type I error is the probability of rejecting a true null hypothesis and that the type II error is the probability of failing to reject a false null hypothesis.

Let us apply this test to the Honolulu Heart Study. The mean age for the population was $\mu = 54.36$. If we did not know this, but guessed that μ was 53, the upper critical point for the distribution under the null hypothesis of 53 would be 53.90, because

$$1.645 = \frac{\bar{x} - 53}{5.5/\sqrt{100}}$$

reduces to $\bar{x} = 53.90$. Figure 9.7 illustrates that if we had randomly arrived at an $\bar{x}$ below 53.90, we would have failed to reject the false H_0 (that $\mu = 53$) β proportion of the time. This β error is represented by the area to the left of $\bar{x} = 53.90$. This area, based on an $\bar{x}$ of 53.90 and an s of 5.5, using a sample of 100, is equal to the area corresponding to

$$Z = \frac{53.90 - 54.36}{5.5/\sqrt{100}} = \frac{-.46}{.55} = -.84$$

Using Table A, we find that $\beta = .20$.

In Figure 9.7 we see that we would have rejected the false H_0 about 80% of the time $(1 - \beta = .80)$. The quantity $1 - \beta$ is referred to as the **power of a test,** which is the probability of rejecting H_0 when H_0 is indeed false. Generally, statisticians try to design statistical tests that have high power; that is, β is small, say, .2 or .1. We can infer from Figure 9.7 that this goal could be accomplished
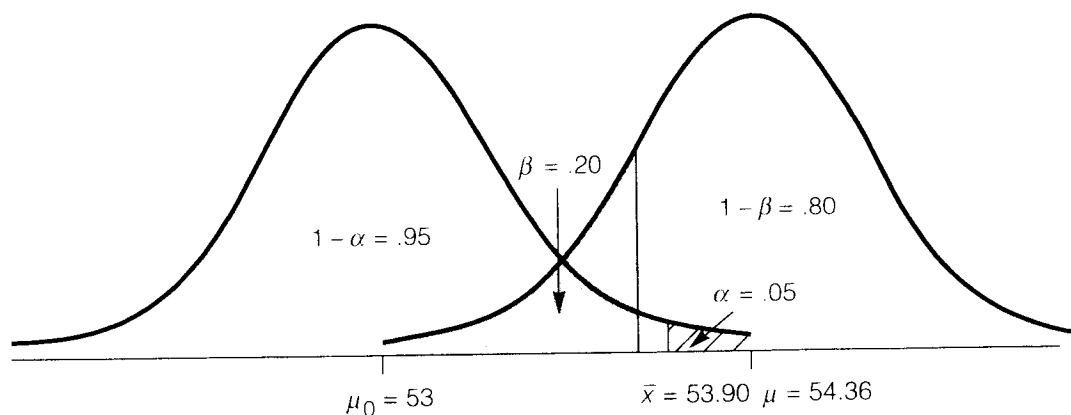
Figure 9.7   Distribution of Sample Means for $\mu_0 = 53$ and $\mu = 54.36$

either by decreasing the significance level (α) from .01 to .05 or by increasing the sample size.

From the foregoing discussion, it should be clear that the α level represents the probability of a type I error, and β, the probability of a type II error. A sort of reciprocal relationship exists between the two types of error. Figure 9.7 suggests that the smaller you choose α to be, the larger β will be. The reason for this is that as the critical region moves farther to the right, more β area is generated to the left of the critical point. The only way to reduce both α and β errors is to reduce the overlap—that is, the area common to the two distributions. This can be done by increasing the sample size, which will reduce $s_{\bar{x}} = s/\sqrt{n}$ and thus narrow the sampling distributions.

.7   TEST OF SIGNIFICANCE OF TWO INDEPENDENT SAMPLE MEANS

We learned in Chapter 8 of the frequent need to compare sample means. As we seldom know the value of σ, we estimate it by s_p (see equation 8.9) and compute the test statistic, which we defined as

$$t = \frac{\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)}{s_p \sqrt{1/n_1 + 1/n_2}} \tag{9.1}$$

with $n_1 + n_2 - 2$ df. Using this test statistic, we compare $\bar{x}_1 - \bar{x}_2$, the difference between the sample means (an estimate of the difference between population means), with $\mu_1 - \mu_2$, the unknown difference between the population means. Because under the null hypothesis the difference between the two means $\mu_1 - \mu_2$ equals zero, in equation 9.1 the expression $\mu_1 - \mu_2$ vanishes.

This situation was illustrated in Example 2 of Chapter 8. Recall that a random sample (n_1) of 25 from a population of joggers provided an estimate of mean maximal VO_2 uptake ($\bar{x}_1$) of 47.5 ml/kg with $s_1 = 4.8$, and for a sample of $n_2 = 26$ from a population of nonjoggers a mean maximal VO_2 uptake of $\bar{x}_2 = 37.5$ ml/kg with an s_2 of 5.1. Is this difference statistically significant or can it be explained by chance? Using the test statistic, we can proceed as follows:

1. $H_0: \mu_1 = \mu_2;\ H_1: \mu_1 \neq \mu_2$.
   Another way of writing $\mu_1 = \mu_2$ is $\mu_1 - \mu_2 = 0$, giving $H_0: \mu_1 - \mu_2 = 0$ and $H_1: \mu_1 - \mu_2 \neq 0$.
2. Significance level $\alpha = .01$.
3. To proceed with the test statistic, we compute s_p by using equation 8.9:

$$s_p = \sqrt{\frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}}$$

$$= \sqrt{\frac{(4.8)^2(24) + (5.1)^2(25)}{25 + 26 - 2}} = \sqrt{\frac{1203.21}{49}}$$

$$= \sqrt{24.56} = 4.96$$

4. The test statistic is computed by using equation 9.1, which is modified only to the extent of dropping $\mu_1 - \mu_2$:

$$
\begin{aligned}
t &= \frac{\bar{x}_1 - \bar{x}_2 - 0}{s_p\sqrt{1/n_1 + 1/n_2}} \\
&= \frac{47.5 - 37.5 - 0}{4.96\sqrt{1/25 + 1/26}} \\
&= \frac{10.00}{1.39} = 7.2
\end{aligned}
\tag{9.2}
$$

5. The critical region for a t with $n_1 + n_2 - 2 = 49$ df is shown in Figure 9.8. This is a two-tailed test, so the t value is found in the column labeled $\alpha = .01/2 = .005$, that is, 2.68 in Table B in the inside back cover.

6. The computed t of 7.2 falls well into the critical region, so we reject the null hypothesis and conclude that joggers have significantly better physical condition than nonjoggers, as judged by their VO_2 uptake.
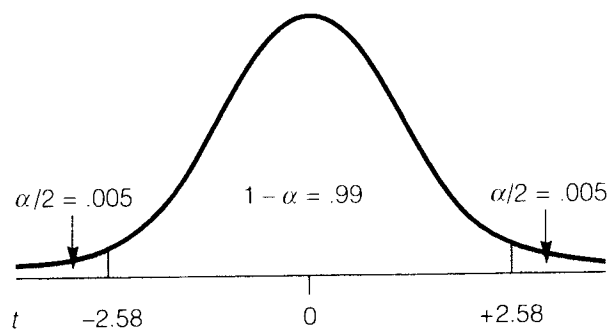
Figure 9.8   Critical Region for a Test Statistic

Those with little experience in statistics are often tempted to use equation 9.2 to test the difference between the two means obtained in a paired t test, as alluded to in Section 8.5. This is a faulty approach because the assumption behind equation 9.2 presupposes two independent samples, whereas the paired t test yields two sample means that are dependent. A desirable way to handle the latter case is to reduce it to a single population statistic (as illustrated by Table 8.1) and apply the following test statistic:

$$t = \frac{d - 0}{s_d / \sqrt{n}} \tag{9.3}$$

with $n - 1$ df. The value $\bar{d}$ is the mean difference between x (before) and x' (after) for each of the cases; s_d is the estimate of the standard deviation of the differences; and zero is used for the difference between the mean before the experiment and the mean after the experiment. Equation 9.3 is also referred to as a *paired t test*.

.8 RELATIONSHIP OF TESTS OF SIGNIFICANCE TO CONFIDENCE INTERVALS

Confidence intervals are determined from Z or t statistics, so you might suspect that the decision reached by use of a significance test would be the same as that reached by use of a confidence interval. And it is indeed the same whenever the hypothesis test is two-tailed. When the significance test was performed on the difference of mean VO_2 uptake between joggers and nonjoggers, it was found that the difference was highly significant. The 99% confidence interval for the difference $\mu_1 - \mu_2$ was 6.42 to 13.58, which did not include the hypothesized mean of zero. Consequently, because zero was not in this interval, we reached the conclusion that the difference was not likely to have occurred by chance alone at the 1% significance level. Both confidence limits being positive, we conclude with 99% confidence that the difference was probably between 6.42 and 13.58, thereby excluding zero as a likely possibility.

Generally, there are two rules to follow in using confidence intervals to determine whether a difference is significant:

1. If a hypothesized difference in means such as $\mu_1 - \mu_2 = 0$ is included in the confidence interval, H_0 is *not* rejected.
2. If the hypothesized difference is *not* included, H_0 is rejected.

So far we have considered tests of significance in which we compare either sample means with population means or the differences between sample means for two groups. It is also possible to compare simultaneously the differences among three or more sample means. The technique for doing this is described in Chapter 10.

.9 SUMMARY TABLE OF INFERENCE FORMULAS

Table 9.1 is a convenient summary of the confidence intervals and test statistics used in testing hypotheses of specific parameters. The confidence intervals are discussed in Chapter 8 for μ and $\mu_1 - \mu_2$ and for π and $\pi_1 - \pi_2$ in Chapter 11.

Table 9.1 Summary Table of Confidence Intervals and Test Statistics for Various Parameters

Parameter	Confidence Interval	Hypothesis	Test Statistic
μ	$(1 - \alpha)\,100\%$ CI for μ $= x \pm Z\dfrac{\sigma}{\sqrt{n}}$ or $= x \pm t\dfrac{s}{\sqrt{n}}$ if σ unknown	$H_0: \mu = \mu_0$	$Z = \dfrac{x - \mu_0}{\sigma/\sqrt{n}}$ $t = \dfrac{x - \mu_0}{s/\sqrt{n}}$ $df = n - 1$
π^*	$(1 - \alpha)\,100\%$ CI for π $= p \pm Z\sqrt{\dfrac{p(1 - p)}{n}}$	$H_0: \pi = \pi_0$	$Z = \dfrac{p - \pi_0}{\sqrt{\dfrac{\pi_0(1 - \pi_0)}{n}}}$
δ	$(1 - \alpha)\,100\%$ CI for δ $= d \pm t\dfrac{s_d}{\sqrt{n}}$	$H_0: \delta = 0$	$t = \dfrac{d - 0}{s_d/\sqrt{n}}$ $df = n - 1$
$\mu_1 - \mu_2$	$(1 - \alpha)100\%$ CI for $(\mu_1 - \mu_2)$ $= x_1 - x_2 \pm Z\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}$ or $= x_1 - x_2 \pm ts_p^{**}\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}$	$H_0: \mu_1 - \mu_2 = 0$	$Z = \dfrac{x_1 - x_2 - 0}{\sigma\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ or $t = \dfrac{x_1 - x_2 - 0}{s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ $df = n_1 + n_2 - 2$
$(\pi_1 - \pi_2)^*$	$(1 - \alpha)\,100\%$ CI for $(\pi_1 - \pi_2)$ $= p_1 - p_2 \pm Z\sqrt{\dfrac{p_1(1 - p_1)}{n_1} + \dfrac{p_2(1 - p_2)}{n_2}}$	$H_0: \pi_1 - \pi_2 = 0$	$Z = \dfrac{p_1 - p_2 - 0}{\sqrt{p'(1 - p')\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$ where $p' = \dfrac{x_1 + x_2}{n_1 + n_2}$

*Discussed in Chapter 11.

$**s_p^2 = \dfrac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - 2}$

Note that the first parameters deal with one population whereas the last two parameters deal with two populations. Furthermore, note that the standard errors of $\pi_1 - \pi_2$ are calculated a little differently for the confidence interval formula than for the test statistic; this will be discussed further in Chapter 11.

.10   SENSITIVITY AND SPECIFICITY

A patient's diagnosis often depends on the outcome of a measurement of a clinical test. Frequently, the measurement has a wide range for both the clinically normal and the diseased states. And because there is no definite dividing line between the normal and the diseased conditions, it is possible that a patient classified as abnormal could indeed be normal, and a patient classified as normal could indeed be abnormal.

To classify an individual as having or not having a certain condition, we need to compare the value of a clinical test to some given cutoff point that divides individuals into normal or abnormal individuals. A clinical value follow-up in the abnormal range suggests that the person has the disease; a value falling in the normal range suggests that the person does not have the disease. Here, as with the rejection of the H_0, it is possible to make two errors:

1. Classifying a person as diseased when one is not (also referred to as a **false positive**)
2. Classifying a person as not diseased when one has the disease (also referred to as a **false negative**)

We can better understand these terms by looking at the following symbolic representation of the results of classification:

		"True" Patient Condition		
		Diseased	Not Diseased	
Result of Test	Disease Present	a	b	$a + b$
	Disease Absent	c	d	$c + d$
		$a + c$	$b + d$	$a + b + c + d$

The false negatives are represented by c, and the false positives are represented by b.

In comparing the effectiveness of different clinical tests or screening tests, we are interested in knowing what their *sensitivities* and *specificities* are. **Sensitivity** is the probability that the clinical test declares those persons positive who have the disease. In terms of the table,

$$\text{Sensitivity} = \frac{a}{a + c}$$

Specificity is the probability that the clinical test declares those persons negative who are without the disease—that is,

$$\text{Specificity} = \frac{d}{b + d}$$

We can see from the table that a relationship exists between these two probabilities and the false negatives and the false positives; that is, the probability of being a false negative is 1 minus the sensitivity and the probability of a false positive is 1 minus the specificity.

■ EXAMPLE 3

A screening program for diabetics used a cutoff point for blood glucose level of 125 mg/100 ml. Those with values above this level were considered diabetics and those below were not. Using the results shown in Table 9.2 for 100 individuals, find the sensitivity and the specificity of this screening test.

$$\text{Sensitivity} = \frac{a}{a + c} = 100 \times \frac{5}{6} = 83.3\% \ (16.7\% \text{ false negative})$$

$$\text{Specificity} = \frac{d}{b + d} = 100 \times \frac{81}{94} = 86.2\% \ (13.8\% \text{ false positive})$$

These results indicate that using a blood glucose cutoff point of 125 mg/100 ml is a procedure with an 83.3% sensitivity and 86.2% specificity; that is, this procedure will declare an average 16.7% individuals not to be diabetics when they

Table 9.2 Outcome of Diabetic Screening Program

	Diabetic	Nondiabetic	
Above 125 mg/100 ml	5	13	18
Below 125 mg/100 ml	1	81	82
	6	94	100

are diabetics, and will declare 13.8% of individuals to be diabetics when they are not. Since sensitivity and specificity are both binomial proportions, we can compute for them the standard errors and confidence intervals, as given in Table 9.1. ∎

Conclusion

Tests of significance are performed to determine the validity of claims regarding the parameters (e.g., μ_1 or $\mu_1 - \mu_2$) of a population. From the nature of each claim, we can decide whether the test should be one-tailed or two-tailed. The decision determines how the null and alternative hypotheses are stated and the manner in which the test is performed. Together with the choice of significance level, the decision defines the critical region. The critical region is the decision-making feature of the test, and the computed test statistic is compared to it. If the value of the test statistic falls in the critical region, we reject the null hypothesis and fail to reject the alternative; if it falls outside the critical region, we fail to reject the null hypothesis and cannot "accept" the alternative. In the former case the evidence supports the claim; in the latter it is insufficient to support the claim. This is equivalent to saying that the result is statistically significant if the P value is small—that is, if the P value is smaller than the value of α. It is possible to commit one of two errors in executing these tests. In rejecting a true null hypothesis we make a type I error (α error), whereas in accepting a false null hypothesis we make a type II error (β error). If we do not wish to define a critical region, it is possible to compute a P value, which indicates the probability of the chance occurrence of this or a larger value of the test statistic when the null hypothesis is true.

Vocabulary List

alternative hypothesis	one-tailed test	test of significance
critical region (rejection region)	power of a test	(hypothesis test)
	P value	test statistic
false negative	sensitivity	two-tailed test
false positive	significance leve	type I error (α error)
nonrejection region	specificity	type II error (β error)
null hypothesis	statistical significance	

Exercises

9.1 What is the critical value for a test of significance in each of the following situations?
   a. One-tailed test, $\alpha = .05$, σ known, $n = 20$
   b. One-tailed test, $\alpha = .05$, σ unknown, $n = 10$
   c. Two-tailed test, $\alpha = .01$, σ unknown, $n = 14$
   d. Two-tailed test, $\alpha = .01$, σ known, $n = 25$
   e. Two-tailed test, $\alpha = .05$, σ unknown, $n = 35$

9.2 In which of the situations in Exercise 9.1 would you use (a) a Z test? (b) a t test? Why?

9.3 For each of the following, state the null (H_0) and alternative (H_1) hypotheses:
   a. Has the average community level of suspended particulates for the month of August exceeded 30 units per cubic meter?
   b. Does mean age of onset of a certain acute disease for schoolchildren differ from 11.5?
   c. A psychologist claims that the average IQ of a sample of 60 children is significantly above the normal IQ of 100.
   d. Is the average cross-sectional area of the lumen of coronary arteries for men ages 40–59, less than 31.5% of the total arterial cross section?
   e. Is the mean hemoglobin level of a group of high-altitude workers different from 16 g/cc?
   f. Does the average speed of 50 cars as checked by radar on a particular highway differ from 55 mph?

9.4 Determine the critical value that would be used to test a hypothesis under the conditions given in each of the following:
   a. $H_0{:}\mu = 220$, $H_1{:}\mu \neq 220$, $\alpha = .05$, $n = 20$, σ known
   b. $H_0{:}\mu \leq 15$, $H_1{:}\mu > 15$, $\alpha = .01$, $n = 35$, σ known
   c. $H_0{:}\mu = 70$, $H_1{:}\mu \neq 70$, $\alpha = .01$, $n = 18$, σ known
   d. $H_0{:}\mu = 120$, $H_1{:}\mu \neq 120$, $\alpha = .05$, $n = 25$, σ unknown
   e. $H_0{:}\mu \geq 100$, $H_1{:}\mu < 100$, $\alpha = .01$, $n = 16$, σ unknown
   f. $H_0{:}\mu \geq 55$, $H_1{:}\mu < 55$, $\alpha = .05$, $n = 49$, σ unknown

9.5 For each of the following situations, choose an α appropriate to the seriousness of the potential error involved should the null hypothesis be rejected when it is actually true.
   a. You wish to decide if a new treatment for pancreatic cancer, known to be a usually fatal disease, is superior to the standard treatment.
   b. The claim is made that the mean income for families of size four is greater than $20,000.

9.6 For each of the parts of Exercise 9.4, decide if you should reject H_0 or fail to reject H_0 according to the corresponding test statistic:
   a. $Z = -1.79$
   b. $Z = 2.01$
   c. $Z = 3.63$
   d. $t = 2.77$
   e. $t = -2.14$
   f. $t = -1.82$

9.7 Boys of a certain age have a mean weight of 85 lb. A complaint is made that in a municipal children's home the boys are underfed. As one bit of evidence, all 25 boys of the given age are weighed and found to have a mean weight of 80.94 lb.
   a. If it is known in advance that the population standard deviation for weights of boys this age is 11.6 lb, what would you conclude regarding the complaint? Use $\alpha = .05$.

b. Suppose the population standard deviation is unknown. If the sample standard deviation is found to be 12.3 lb, what conclusion regarding the complaint might you draw? Use $\alpha = .05$.

9.8 In Table 3.1, the mean systolic blood pressure is 130 mmHg and the variance is 448. Is this an indication that the group is significantly different from the standard if the population standard is known to be 120 mmHg? Test at $\alpha = .05$.

9.9 a. Calculate the P value for each case in Exercises 8.3, 8.4, 8.5, 9.7, 9.8, and 9.17.
b. For each test, what do the P values tell you about statistical significance?
c. Do your answers to (b) agree with the decisions and conclusions you made in each exercise? Why? Why not?

9.10 a. State the value of the type I error for each case in Exercises 8.3, 8.4, 8.5, 9.7, 9.8, and 9.17.
b. What does the type I error tell you?
c. What does the type II error tell you?
d. From the information stated in the problems, are you able to state the type II errors?
e. If in any given problem you should decide to decrease the type I error (say from .05 to .01), what would happen to the type II error?
f. What is usually done to avoid type II errors?
g. What could you do to reduce both types of error simultaneously?

9.11 In Table 2.2, the means and standard deviations of some subgroups of the sample are as follows:

	Mean	Standard Deviation	n
Vegetarians	72.9	11.7	40
Nonvegetarians	73.5	11.4	43
Males	74.9	12.0	38
Females	71.8	11.0	45

Is there a significant difference in the mean diastolic blood pressures, at $\alpha = .05$, between
a. vegetarians and nonvegetarians?
b. males and females?

9.12 Birth lengths of male and female infants in a small clinic gave the following results:

Group	Sample Size	x (cm)	s (cm)
Males	12	52.2	8.6
Females	9	50.7	9.5

Assuming normally distributed populations with equal variances, do these data justify the conclusion, at $\alpha = .05$, that the mean birth length is greater for males than for females? Also calculate the P value for the computed t.

9.13 For Exercise 8.2, determine whether the mean hemoglobin level of the group of 25 men is significantly different from $\mu = 15$ at the $\alpha = .05$ level.

9.14 Ten experimental animals were subjected to conditions simulating disease. The number of heartbeats per minute, before and after the experiment, were recorded as follows:

Animal	Heartbeats per Minute Before	After	d	Animal	Heartbeats per Minute Before	After	d
1	70	115	45	6	120	115	−5
2	84	128	44	7	110	110	0
3	88	146	58	8	67	140	73
4	110	171	61	9	79	131	52
5	105	158	53	10	86	157	71

Do these data provide sufficient evidence to indicate that the experimental condition increases the number of heartbeats per minute? Let $\alpha = .05$. Also calculate the P value for the computed t.

9.15 Blood samples from 10 persons were sent to each of two labs for cholesterol determinations.

Subject	Serum Cholesterol (mg/ml) Lab 1	Lab 2
1	296	318
2	268	287
3	244	260
4	272	279
5	240	245
6	244	249
7	282	294
8	254	271
9	244	262
10	262	285
Σx	2,606	2,750
Σx^2	682,316	760,706
s_x	18.83	22.25

Is there a statistically significant difference (at the $\alpha = .01$ level) in the cholesterol levels reported by lab 1 and lab 2?
a. Should one use the pooled t test or the paired t test to answer this question?
b. Perform the test you chose for (a) and answer the question.

    c. Perform the test you did *not* choose for (a) and compare the result with (b). What do you observe?

    d. Determine the P values for both (b) and (c) and compare them. Discuss the relationship of the P values to what you have already concluded about the two t tests.

9.16 If in Exercise 8.4 you found for the group of 25 men a mean of .35 g/6 hr, would you conclude that this group was significantly different from the standard group at the $\alpha = .01$ level?

9.17 The mean diastolic blood pressure in Table 2.2 is 73 mmHg with a standard deviation of 11.6 mmHg. For an α of .01, test whether the mean blood pressure of this group is significantly greater than 70.

9.18 The mean weight of the sample of 100 persons from the Honolulu Heart Study was 64 kg. If the ideal weight is known to be 60 kg, is the group significantly overweight? Assume $\sigma = 10$ kg and $\alpha = .05$.

9.19 a. For the data in Exercise 8.8, indicate at an α of .05 whether the mean cholesterol level of the vegetarian group is significantly lower than that of the nonvegetarians.

    b. Compute a P value for the test statistic.

9.20 Describe the difference between the H_0 and H_1.

9.21 a. What assumptions regarding the difference of two means are made in performing the t test?

    b. What is the basis for pooling the sample variances when testing the difference between two population means?

9.22 a. What is the basis for being able to use confidence intervals to perform a test of a hypothesis?

    b. What are the rules governing its use?

9.23 a. Determine whether there is a significant difference between the mean systolic blood pressures of smokers and nonsmokers at the $\alpha = .05$ level using the data from Exercise 8.12.

    b. Is the decision reached by use of the confidence interval the same as that reached in (a)? Explain why or why not?

9.24 Determine whether there is a significant difference at the $\alpha = .01$ level in the mean weight gains of the two diets described in Exercise 8.14.

9.25 a. Using the data of Exercise 8.15, determine whether there is a significant difference in the mean serum cholesterol levels of men and women at the $\alpha = .05$ level.

    b. Did you reach the same decision in part (a) that you would have reached if you had used the 95% confidence interval for $(\mu_1 - \mu_2)$?

9.26 a. Using the data of Exercise 8.16, determine whether there is a significant difference in the mean hemoglobin levels of white and black women at the $\alpha = .05$ level.

    b. Did you reach the same decision you would have reached if you had used the 95% confidence interval for $(\mu_1 - \mu_2)$?

9.27 A study was conducted using 139 undergraduates at a large private university who volunteered to participate in this research as partial fulfillment of a course requirement. One of the items studied was the maximum daily amount of alcohol consumed in the last month. Based on the data in the following table, are there differences between males and females in the maximum amount of alcohol consumed in any one day in the past month?

Maximum Daily Quantity of Alcohol Consumed in the Last Month	
Men	Women
Mean = 8.2	Mean = 5.6
$s = 5.9$	$s = 5.7$
$n = 54$	$n = 85$

NOTE: These data were extrapolated and based on Carey and Correia (1997).

a. Write your null and research hypothesis using the correct statistical notation.
b. What is your critical value at a .05 level of significance?
c. Are these groups independent or dependent? Explain.
d. What is your calculated t value?
e. State your conclusions. Be specific.
f. Calculate the 95% confidence intervals.
g. Refer to Chapter 2,"Populations and Samples." What concerns would you have about generalizing these results to all college students? Identify as many concerns as you can.

10 Analysis of Variance

Chapter Outline

10.1 Function of ANOVA
Discusses the general usefulness of ANOVA (analysis of variation) for comparing means of several groups

10.2 Rationale for ANOVA
Explains how ANOVA utilizes a comparison of variations between and within groups by means of an F ratio

10.3 ANOVA Calculations
Shows the equations for the various sources of variation

10.4 Assumptions
Describes the assumptions necessary for performing the tests of hypotheses—independence, normality, and homogeneity of variance

10.5 Application
Uses the testing of the hypothesis of equality of mean birthweights among the infants of three groups of mothers classified by smoking status to test the one-way ANOVA classification

10.6 Tukey's HSD Test
Illustrates Tukey's HSD test, a procedure for performing multiple comparison tests

10.7 Randomized Block Design
Describes a randomized block design in which blocks are divided into experimental units

Learning Objectives

After studying this chapter, you should be able to

1. Indicate the circumstances that call for an ANOVA rather than a t test

2. Set up an ANOVA table that partitions the total sum of squares into between-group and within-group sums of squares

3. Compute the F ratio and its appropriate degrees of freedom

4. List the two assumptions that need to be made to perform an ANOVA

5. Indicate the type of hypothesis that can be tested with an ANOVA

6. Find the critical region for an F-ratio test

7. Indicate the reason for performing multiple range tests

8. Describe how to apply Tukey's multiple comparison procedure

9. Describe an example of a randomized block design

10.1   FUNCTION OF ANOVA

Analysis of variance (**ANOVA**) is a powerful method of analyzing differences among a number of groups. It deals with the comparison of means from several groups. In Chapter 9 we discussed the technique for testing the significance of the difference between means for two groups. But how do you determine, for instance, whether there are significant differences in birthweight among three groups of infants—the first group born to nonsmoking mothers, the second to light-smoking mothers, and the third to heavy-smoking mothers?

It is possible to perform t tests between the means of each pair of groups and determine which pairs differ significantly within the pairs. But this approach presents a number of difficulties—the choice of a proper significance level for "overtesting," the numerous tests needed if many groups are involved, and the lack of one overall measure of significance for the differences among the means.

ANOVA is able to handle these problems elegantly. Because the results obtained with ANOVA for two groups is identical to the results obtained with a t test, it is fair to say that ANOVA is an extension of the t test to handle more than two independent groups.

For the birthweight example, the null hypothesis being tested is

$$H_0: \mu_1 = \mu_2 = \mu_3$$

the alternative hypothesis, H_1, being that H_0 is not true; that is, either one of the means is not equal to the others or none of them are equal to one another. The three smoking-status groups would be commonly referred to as the treatment groups, with smoking exposure considered as the "treatment." The theoretical basis for performing this test is the partitioning of the available variance of all observations into two sources of variation—variation *between** the group means and variation *within* each of the groups. The sampling distribution used for testing these means is not the t distribution but rather the **F distribution** (named in honor of the celebrated R. A. Fisher, who developed the F statistic).

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*When comparing more than two groups that are not reciprocally related, the term "among" is grammatically preferable to "between." In the present context, the somewhat ungrammatical but traditional use of "between" is due to the work of some pioneer statisticians.

).2 RATIONALE FOR ANOVA

Analysis of variance is unique in that it compares two different estimates of the population variance to test a hypothesis concerning the population mean. One of these estimates is **within-group variance,** which is simply the sum of the variances of each of the groups. It is analogous to the s_p^2 used in t tests, extended to the sum of the sample variances of more than two groups. It is called *within-group* variance because it is the collective variance of all observations within each group. By convention, within-group variance is denoted by s_w^2. The other estimate of variance is **between-group variance,** which measures the variation between the means of the various groups and is denoted by s_b^2. The within-group variance and the between-group variance are also referred to as the **mean squares** or **MS.** The terms within-group variance and MS within are interchangeable as are the terms between-group variance and MS between. The ANOVA tables used in this chapter are labeled MS or mean squares. Using mathematical statistics, we can demonstrate that the between-group variance is equal to the within-group variance if the means of each group are equal; that is, there is *no* treatment effect.

With this knowledge, we can perform a test of the hypothesis of equality of means by comparing the ratio of the two variance estimates, s_b^2/s_w^2. If the two variances are indeed equal, the ratio s_b^2/s_w^2 should be approximately 1. Because we are dealing with s^2, an estimate of σ^2, the ratio will sometimes be greater and sometimes smaller than 1 even if the hypothesis of equal means is true. The ratio s_b^2/s_w^2 follows the F distribution and is illustrated in Figure 10.1.

In fact, there is a family of F distributions, one for each pair of degrees of freedom. The F statistic follows a skewed distribution, with two sets of degrees of freedom. The variance estimate s_b^2 has $k - 1$ df, where k is the number of groups; s_w^2 has $k(n_i - 1)$, where n_i is the number of observations in each group. The number of observations per group do not have to be equal. The table in Appendix B gives critical values for the F distribution. Note that separate tabulations are provided for $\alpha = .05$ and $\alpha = .01$. For example, the critical F values for 2 and 30 df are 3.32 for an α of .05 (Figure 10.1) and 5.32 for an α of .01.
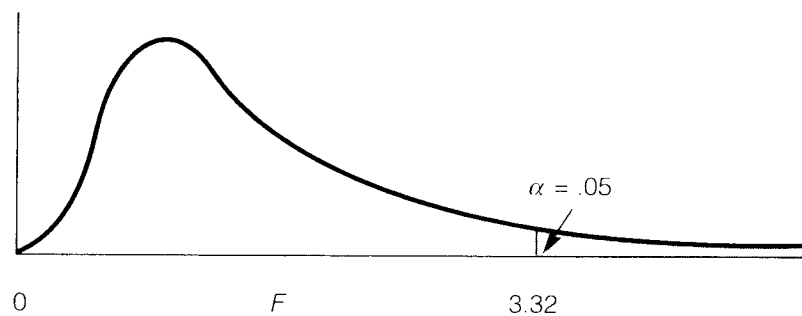
Figure 10.1 Critical Value of $F_{2,30} = 3.32$, $\alpha = .05$

10.3   ANOVA CALCULATIONS

We need a systematic procedure for computing ANOVA. To illustrate the procedure, we use data from the general case shown in Table 10.1. We can see that there can be an unequal number of observations for each of the k groups. The formulas given next accommodate this situation. The observations within each group are indicated with **double notation,** where the first subscript indicates the group number and the second subscript indicates the observation in that group: for example, x_{12} is the second observation in group 1. By extension, the jth observation in the ith group is indicated by x_{ij}. The mean for group 1 is denoted by x_1 and is shown by

$$\bar{x}_1 = \sum_{j=1}^{n_1} \frac{x_{1j}}{n_i} = \frac{x_1}{n_1} \tag{10.1}$$

The sum of all observations is given by

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} \tag{10.2}$$

The overall mean is obtained by dividing the total of all observations by the total number of observations,

$$N = \sum_{i=1}^{k} n_i$$

Table 10.1 Symbolic Representation of Data in a One-Way Analysis of k Groups, with Equal Number of Observations per Group

	Group	
	1	2		i		k
	x_{11}	x_{21}	$\cdots$	x_{i1}	$\cdots$	x_{k1}
	x_{12}	x_{22}	$\cdots$	x_{i2}	$\cdots$	x_{k2}
	x_{13}	x_{23}	$\cdots$	x_{i3}	$\cdots$	x_{k3}
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	x_{1j}	x_{2j}	$\cdots$	x_{ij}	$\cdots$	x_{kj}
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	x_{1n}	x_{2n_2}	$\cdots$	x_{in}	$\cdots$	x_{kn_k}
Total	$\sum x_{1j}$	$\sum x_{2j}$	$\cdots$	$\sum x_{ij}$	$\cdots$	$\sum x_{kj}$	$\sum\sum x_{ij}$ (grand total)
Mean	x_1	x_2	$\cdots$	x_i	$\cdots$	x_k	x (grand mean)

where n_i is the number of cases per group. Therefore, the overall mean is

$$x = \frac{\Sigma\Sigma x_{ij}}{N}$$

Now the **between-group sum of squares** (SS_b)—that is, the sum of the squared deviations between groups—is needed for computing the between-group variance. This SS_b can be obtained from Table 10.1 by use of

$$SS_b = \sum_{i=1}^{k} n_i[(\bar{x}_i - \bar{x})^2]$$

$$= n_1(\bar{x}_1 - \bar{x})^2 + n_2(\bar{x}_2 - \bar{x})^2 + \cdots + n_k(\bar{x}_k - \bar{x})^2$$

(10.3)

The **within-group sum of squares**, SS_w, needed for computing the within-group variance, can be obtained by use of

$$SS_w = \sum_{i=1}^{k} \left[\sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 \right]$$

(10.4)

The total sum of squares, SS_t, which measures the amount of variation about the overall mean, is the sum of the squared deviations of each x_{ij} from the overall mean. To obtain it, we use

$$SS_t = \sum_{i=1}^{k} \left[\sum_{i=1}^{n_i} (x_{ij} - \bar{x})^2 \right]$$

A little algebraic manipulation shows that $SS_t = SS_b + SS_w$; that is,

$$\sum_{i=1}^{k} \left[\sum_{j=1}^{n_i} (x_{ij} - \bar{x})^2 \right] = \sum_{i=1}^{k} n_i \left[(\bar{x}_i - \bar{x})^2 \right] + \sum_{i=1}^{k} \left[\sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 \right]$$

(10.5)

which suggests that the total variation of observations from the overall mean can be partitioned into two parts: the variation of the sum of squares between groups and that of the sum of squares within groups. The total number of degrees of freedom ($\Sigma n_i - 1$) is equal to the sum of the between group ($k - 1$) plus the within group ($\Sigma n_i - k$).

From equations 10.3 and 10.4 it follows that the F statistic used to test the hypothesis of equality of means is

$$F_{k-1,(\Sigma n_i - k)} = \frac{s_b^2}{s_w^2} = \frac{SS_b/(k-1)}{SS_w/(\Sigma n_i - k)} = \frac{\Sigma\Sigma(\bar{x}_i - \bar{x})^2/(k-1)}{\Sigma\Sigma(x_{ij} - \bar{x}_i)^2/(n-k)}$$

To complete an ANOVA table, we usually calculate only SS_t and SS_b. The SS_w is obtained by $SS_w = SS_t - SS_b$. To calculate we use

$$SS_t = \sum_{i=1}^{k}\sum_{j=1}^{n} x_{ij}^2 - \frac{\left[\sum_{i=1}^{k}\sum_{i=1}^{n_i} x_{ij}\right]^2}{\sum_{i=1}^{k} n_i} \qquad (10.6)$$

and

$$SS_b = \sum_{i=1}^{k} n_i \bar{x}_i^2 - \frac{\left[\sum_{i=1}^{k}\sum_{i=1}^{n_i} x_{ij}\right]^2}{\sum_{i=1}^{k} n_i} \qquad (10.7)$$

10.4   ASSUMPTIONS

To perform tests of hypotheses we need to make two assumptions:

1. The observations are independent; that is, the value of one observation is not correlated with that of another.
2. The observations in each group are normally distributed, and the variance of each group is equal to that of any other group; that is, the variances of the various groups are **homogeneous.**

We should point out that ANOVA is a **robust technique,** insensitive to departures from normality and homogenity, and is particularly so if the sample sizes are large and nearly equal for each group.

10.5   APPLICATION

Let us return now to the question posed at the beginning of this chapter: Is there a significant difference in birthweight among three groups of infants classified by the smoking status of the mothers? The analysis procedure would be as follows:

1. H_0: $\mu_1 = \mu_2 = \mu_3$.
   H_1: That one or more mean is different from the others.

2. Test statistic: $F = s_b^2/s_w^2$ for $k - 1$, $\Sigma n_i - k$ df, or $F = \text{MS}_b/\text{MS}_w$.

3. Rejection region: We reject H_0 if the computed F statistic is greater than the tabulated value for α with the given degrees of freedom.

Using the observations of Table 10.2 and the equations of Table 10.3, we can set things up in a conventional ANOVA table. Note that in Table 10.3 the **mean squares** are the sums of squares divided by their respective degrees of freedom.

Table 10.2 Infant Birthweights (grams) and Means Classified by Smoking Status of Three Groups of Mothers

	Smoking Status			
	None	1 Pack/Day	1 + Pack/Day	
Subject	1	2	3	
1	3,515	3,444	2,608	
2	3,420	3,827	2,509	
3	3,175	3,884	3,600	
4	3,586	3,515	1,730	
5	3,232	3,416	3,175	
6	3,884	3,742	3,459	
7	3,856	3,062	3,288	
8	3,941	3,076	2,920	
9	3,232	2,835	3,020	
10	4,054	2,750	2,778	
11	3,459	3,460	2,466	
12	3,998	3,340	3,260	
$\sum\limits_{j}^{n_i} x_{ij}$	43,352	40,351	34,813	$\Sigma\Sigma x_{ij} = 118{,}516$ (grand total)
$x_{i\cdot}$	3,613	3,363	2,901	$x = 3{,}292$ (grand mean)

Table 10.3 ANOVA Table for a One-Way Classification with an Unequal Number of Observations per Group

Source of Variation	Sum of Squares	df	Mean Squares	F ratio
Between	$\text{SS}_b = \Sigma n_i(x_i - x)^2$	$k - 1$	$\text{MS}_b = \dfrac{\text{SS}_b}{k - 1}$	$F_{k-1,\,\Sigma n_i - k} = \dfrac{\text{MS}_b}{\text{MS}_w}$
Within	$\text{SS}_w = \text{SS}_t - \text{SS}_b$	$\Sigma n_i - k$	$s_p^2 = \dfrac{\text{SS}_w}{\Sigma ni - k}$	
Total	$\text{SS}_t = \sum\limits_{i=1}^{k}\sum\limits_{j=1}^{n}[(x_{ij} - x)^2]$	$\Sigma n_i - 1$		

Using equation 10.6, we have

$$SS_t = \Sigma\Sigma\, x_{ij}^2 - \frac{(\Sigma\Sigma x_{ij})^2}{\Sigma n_i}$$

$$= 398{,}915{,}214 - \frac{(118{,}516)^2}{36}$$

$$= 8{,}747{,}373.556$$

and using equation 10.7, we obtain

$$SS_b = \Sigma n_i\,\bar{x}_i^2 - \frac{(\Sigma\Sigma x_{ij})^2}{\Sigma n_i}$$

$$= \left[12(3613)^2 + 12(3363)^2 + 12(2901)^2 \right] - \frac{(118{,}516)^2}{36}$$

$$= 3{,}184{,}227.556$$

The within sum of squares is, therefore,

$$SS_w = SS_t - SS_b$$

$$= 8{,}747{,}374.0 - 3{,}184{,}227.4$$

$$= 5{,}563{,}146.6$$

Now we are able to set up the ANOVA table for our illustration of the effect of maternal smoking (Table 10.4). From this table we can see that the computed F ratio is greater than the tabulated value of $F_{2,33} = 3.29$. This indicates that at least one of the means is significantly different from the others—that is, that maternal smoking appears to be associated with infant birthweight. To find out which means are significantly different, we may be tempted to perform a number of multiple t tests between the various pairs of means. But it would be inappropriate to do so unless we wanted to know whether there was a significant difference between the nonsmokers and the heavy smokers before seeing the results. Multiple t tests are inappropriate because the probability of incorrectly rejecting the hypothesis increases with the number of t tests performed. So even

Table 10.4 ANOVA for Infant Birthweight Classified by Mother's Smoking Status

Source	SS	df	MS	F
Between	3,184,227.5	2	1,592,113.7	9.44
Within	5,563,146.6	33	168,580.2	
Total	8,747,374.1	35		

though we may be performing a test of significance at $\alpha = .05$, the actual α level is, in effect, made considerably higher. A number of *multiple comparison* procedures have been proposed by statisticians. One that is fairly easy to use was developed by Tukey (1968) and is discussed in the following section.

0.6   TUKEY'S HSD TEST

Tukey's HSD (honestly significant difference) **test** is used to test the hypothesis that all possible pairs of means are equal. Tukey's HSD can only be used if there are an equal number of observations in each group. To perform this **multiple comparison test,** we select an overall significance level, α, which denotes the probability that one or more of the null hypotheses is false. The HSD value is then computed and all differences are compared to it. Those pairs whose differences exceed the HSD are considered significantly different. The formula for computing HSD is

$$\text{HSD} = q(\alpha, \ k, \ N - k)\sqrt{\frac{\text{MS}_w}{n}} \qquad (10.8)$$

where α is the selected significance level; k, the number of groups; N, the total number of observations; n, the number of observations per treatment group; MS_w, the within-mean-square error term; and q is obtained from the table in Appendix C.

■ **EXAMPLE 1**

To determine which of the pairs of groups in Table 10.2

$$\bar{x}_1 - \bar{x}_2 = 3613 - 3363 = 250$$

$$\bar{x}_2 - \bar{x}_3 = 3363 - 2901 = 462$$

$$\bar{x}_1 - \bar{x}_3 = 3613 - 2901 = 712$$

is significantly different, we compute the HSD test. Using $\alpha = .05$, $k = 3$, and $N - k = 36 - 3 = 33$, we find from Appendix C that q is about 3.48. From Table 10.4, MS_w is 168,580 and from Table 10.2, $n = 12$. Therefore,

$$\text{HSD} = 3.48\sqrt{\frac{168,580}{12}}$$

$$= 3.48(118.5) = 412$$

Because $\bar{x}_2 - \bar{x}_3$ and $\bar{x}_1 - \bar{x}_3$ exceed 412, we conclude that there is a significant difference between the birthweight of infants of mothers who do not smoke

versus those who smoke 1+ pack/day and between the birthweight of infants of mothers who smoke 1 pack/day versus 1+ pack/day. The difference in birthweight of infants of mothers who did not smoke versus those who smoked only 1 pack/day was not significantly different. ■

We have discussed ANOVA with unequal number of observations per treatment. The equations, however, will also accommodate unequal numbers of treatments. So far, we have considered only the one-way ANOVA classification. It is possible to work with two-way, three-way, or multiple-way classifications as well. For example, a two-way ANOVA might consider four treatment groups for each sex group, with the second classification being by sex. This method is discussed in the next section.

10.7   RANDOMIZED BLOCK DESIGN

A **randomized block design** is a design in which homogeneous blocks are divided into experimental units to which the treatments are assigned in a random fashion. The purpose of this design is to remove from the error term the variation due to the blocks. Each block has one experimental unit for each treatment and each treatment is represented in each block. Table 10.5 shows the layout of the data from a study that used a randomized block design. (Note that here we use double notation, which facilitates the handling of the formulas). There are k **treatment effects** (effects due to some stimulant) and n blocks. Blocks can be homogeneous subgroups stratified on age, weight, SES (socioeconomic status) group, or other factors. One of the first things usually done is to observe what the treatment and block means are. They are computed in the following fashion. The mean for the first treatment is given by

$$\bar{x}_{1\cdot} = \sum_{j=1}^{n} \frac{x_{ij}}{n} = \frac{x_{1\cdot}}{n}$$

Table 10.5 Symbolic Representation of Values for the Randomized Block Design with k Treatment and n Blocks

Blocks	Treatments 1	2	3	...	k	Total	Mean
1	x_{11}	x_{12}	x_{13}	...	x_{1k}	$x_{1\cdot}$	$\bar{x}_{1\cdot}$
2	x_{21}	x_{22}	x_{23}	...	x_{2k}	$x_{2\cdot}$	$\bar{x}_{2\cdot}$
3	x_{31}	x_{32}	x_{33}	...	x_{3k}	$x_{3\cdot}$	$\bar{x}_{3\cdot}$
⋮	⋮	⋮	⋮		⋮	⋮	⋮
n	x_{n1}	x_{n2}	x_{n3}	...	x_{nk}	$x_{n\cdot}$	$\bar{x}_{n\cdot}$
Total	$x_{\cdot1}$	$x_{\cdot2}$	$x_{\cdot3}$	...	$x_{\cdot k}$	$x_{\cdot\cdot}$	
Mean	$\bar{x}_{\cdot1}$	$\bar{x}_{\cdot2}$	$\bar{x}_{\cdot3}$	...	$\bar{x}_{\cdot k}$		$\bar{x}_{\cdot\cdot}$

and the mean for the first block is given by

$$x_{\cdot 1} = \sum_{i=1}^{k} \frac{x_{i1}}{n} = \frac{x_{\cdot 1}}{k}$$

The sum of all the observations is given by

$$\sum_{i=1}^{k} \sum_{j=1}^{k} x_{ij} = x_{\cdot\cdot}$$

Note that we are assuming we have a balanced design; that is, each block has k treatments and each treatment has n blocks.

■ EXAMPLE 2

Let us look again at the relationship between maternal smoking and infant birthweight, but taking the mother's weight into account. We will look at the same three treatment groups as before: mothers who did not smoke, those who smoked up to 1 pack/day and those who smoked 1+ pack/day. The six weight groups, in increments of 5 kg, are the six blocks. Technically, this design assumes that the smoking "treatment level" was assigned randomly to the pregnant women in a particular block (weight group). Such an assignment, however, was not the case. The data are shown in Table 10.6.

From the means in the table, we can see that there is an inverse relationship between maternal smoking and infant birthweight; that is, the means for the three smoking groups decrease with an increased level of smoking. There is also a direct relationship between prepregnancy weight and infant birthweight; that is, the birthweights increase as the mother's weight increases. To determine whether there is a significant treatment (smoking) effect after we remove the

Table 10.6 Infant Birthweight (grams) and Means Classified by Maternal Smoking Status and Prepregnancy Weight Group

Blocks Group (kg)	Treatments			Total	Mean
	None	1 Pack/Day	1 + Pack/Day		
45–49	3,175	2,750	1,730	7,655	2,552
50–54	3,232	2,835	2,466	8,533	2,844
55–59	3,240	3,062	2,509	8,811	2,937
60–64	3,420	3,076	2,608	9,104	3,035
65–69	3,459	3,340	2,778	9,577	3,192
70–74	3,515	3,416	2,920	9,851	3,284
Total	20,041	18,479	15,011	53,531	
Mean	3,340	3,080	2,502		2,974

variation due to blocks (prepregnancy weight), we need to prepare an ANOVA table. As in the one-way ANOVA, the total sum of squares can be partitioned into three parts: the effect due to blocks, that due to treatment, and a residual part similar to the within term we saw before—that is,

$$SS_t = SS_b + SS_{tr} + SS_r$$

The formulas for these are

$$SS_t = \Sigma\Sigma(x_{ij} - \bar{x}_{..})^2 = \Sigma\Sigma x_{ij}^2 - CT$$

$$SS_b = \Sigma\Sigma(\bar{x}_{i.} - \bar{x}_{..})^2 = \Sigma\Sigma\bar{x}_{i.}^2 - CT$$

$$SS_{tr} = \Sigma\Sigma(\bar{x}_{.j} - \bar{x}_{..})^2 = \Sigma\Sigma\bar{x}_{.j}^2 - CT$$

$$SS_r = \Sigma\Sigma(x_{ij} - \bar{x}_{.j} - \bar{x}_{i.} + \bar{x}_{..})^2$$

CT, the "correction term," is given by

$$CT = \frac{(\Sigma x_{ij})^2}{kn} = \frac{x_{..}^2}{kn}$$

We can apply these formulas to the data in Table 10.6 and obtain

$$CT = \frac{\Sigma\Sigma x_{..}^2}{kn} = \frac{(53,531)^2}{3(6)} = \frac{2,865,567,961}{18} = 159,198,220$$

$$SS_t = \sum_{i=1}^{k}\sum_{j=1}^{n} x_j^2 - CT$$
$$= 162,716,841 - 159,198,220 = 3,518,621$$

$$SS_b = \sum_{i=1}^{k}\sum_{j=1}^{n} x_{.j}^2 - CT = k(x_{.1}^2 + \cdots + x_{.6}^2) - CT$$
$$= 3(2552^2 + \cdots + 3284^2) - CT$$
$$= 3(53,411,754) - 159,198,220$$
$$= 1,037,042$$

$$SS_{tr} = \sum_{i=1}^{k}\sum_{j=1}^{n} \bar{x}_{i.}^2 - CT = n(x_{1.}^2 + \bar{x}_{2.}^2 + x_{3.}^3) - CT$$
$$= 6(3340^2 + 3080^2 + 2502^2) - CT$$
$$= 6(26,902,004) - 159,198,220$$
$$= 2,213,804$$

$$SS_r = SS_t - SS_b - SS_{tr}$$
$$= 3{,}518{,}612 - 1{,}037{,}042 - 2{,}213{,}804$$
$$= 267{,}766$$

The degrees of freedom are also partitioned, as follows:

Total = blocks + treatments + residual

$$kn - 1 = (n - 1) + (k - 1) + (n - 1)(k - 1)$$

For our example, these would be

$$18 - 1 = (6 - 1) + (3 - 1) + (6 - 1)(3 - 1)$$
$$17 = 5 + 2 + 10$$

The layout for the ANOVA table for the randomized blocks design is shown in Table 10.7.

Because we are interested in knowing whether there is a treatment (maternal smoking) effect on infant birthweight after removing the variation due to prepregnancy weight, we proceed as follows:

1. State the H_0: There is no treatment (smoking) effect.
2. We calculate the F ratio using the formula from Table 10.7. If the H_0 is true, both $MS(SS_{tr})$ and $MS(SS_r)$ are estimates of σ^2. Therefore, the F ratio should be about 1.0.
3. If the H_0 is true, the quantity

$$\frac{MS(SS_{tr})}{MS(SS_r)}$$

Table 10.7 ANOVA Table for the Randomized Complete Block Design

Source of Variation	Sum of Squares	df	MS	F ratio
Treatments	SS_{tr}	$k - 1$	$\dfrac{SS_{tr}}{k - 1}$	$\dfrac{MS(SS_{tr})}{MS(SS_r)}$
Blocks	SS_b	$n - 1$	$\dfrac{SS_b}{n - 1}$	
Residual	SS_r	$(k - 1)(n - 1)$	$\dfrac{SS_r}{(k - 1)(n - 1)}$	
Total	SS_t	$kn - 1$		

should follow an F distribution with $(k - 1)$ and $(k - 1)(n - 1)$ degrees of freedom. If the computed value of F is greater than the critical value from the F table, we reject the H_0.

We can now prepare the ANOVA table and reach a decision:

Source	SS	df	MS	F
Treatments	2,213,804	2	1,106,902	41.3
Blocks	1,037,042	5	207,408	
Residual	267,766	10	26,777	
Total	3,518,621	17		

Our computed value of $F = 41.3$ is greater than the critical 1% $F_{2,10} = 7.56$, so we reject the H_0 of no treatment (smoking) effect at the $\alpha = .01$ level. To find out which pairs of means are significant, we could apply Tukey's HSD test. ∎

To perform the test of H_0 we need to make the following assumptions:

1. The observations, x_{ij}, are normally distributed.
2. The treatment effects, the block effects, and the residuals $(x_{ij} - x_{i.} - x_{.j} + x_{..})$ are independent and have the same variances.

The ANOVA technique is quite robust to any violations of these assumptions. Therefore, the results are still valid even when the assumptions are not strictly met. If the violations are considerable, we can frequently remedy the situation by transforming the x_{ij} by taking a log, square root, or reciprocal of them.

Because this is an introductory level text, the presentation of the ANOVA technique is necessarily brief. To learn more about the procedure, readers should consult such textbooks on experimental design as Snedecor (1956) or Steel and Torrie (1980).

Conclusion

The analysis of variance is so named because its test procedure is based on a comparison of the estimate of the between-group variance to the estimate of the within-group variance. These two estimates of σ^2 are obtained by partitioning the overall variance. An F statistic is used to determine the critical region for the test. If the computed F ratio falls in the critical region, we conclude that at least one of the means is significantly different from the others. To determine which

specific pairs of means are significant, we utilize a multiple range test, not multiple t tests. To test the hypothesis, we must assume independence of observations, normality of each group, and homogeneous variances. An important interpretation of ANOVA is that it tests whether there is a treatment effect, where the treatment is drug dosage, smoking exposure, or some other factor.

In this chapter we discussed the one-way classification of variance. To be able to account for the many possible sources of variation in a particular experiment, you may wish to perform a two-way or a three-way ANOVA.

Vocabulary List

ANOVA	homogeneous variances	robust technique
between-group sum of squares	mean squares	treatment effect
	multiple comparison	Tukey's HSD test
between-group variance	tests	within-group sum of
double notation	randomized block	squares
F distribution	design	within-group variance

Exercises

10.1 A survey was done in a community in which residents were asked if they felt that family planning counseling was needed in the community. The tabulation in the accompanying table gives the opinions and the number of children of the respondents.

Determine whether there is a difference in mean number of children of respondents.
a. Give the null hypothesis.
b. Construct an ANOVA table.
c. State your results and conclusions.

	Great Need	Some Need	No Need	
	0	10	17	
	1	5	10	
	3	7	9	
	4	3	3	
Number of children	2	9	15	
	1	8	10	
	3	7	11	
	0	9	10	
	1	10	9	
	2	9	8	
Σx	17	77	102	196 (grand total)
x	1.7	7.7	10.2	6.53 (grand mean)

10.2 Five samples were taken randomly from each blood type, and the white cell counts were noted to be as follows:

	Blood Type				
	A	B	AB	O	
White cell counts	5,000	7,000	7,000	5,325	
	5,500	8,000	7,125	7,985	
	8,000	5,000	9,000	6,689	
	10,000	9,900	9,235	9,321	
	7,735	6,342	7,699	6,666	
Σx	36,235	36,242	40,059	35,986	148,522 (grand total)
x	7247.0	7248.4	8011.8	7197.2	7426.1 (grand mean)

Are the four blood types the same with respect to white cell counts?

10.3 Seven samples of individuals were selected randomly from three communities. The ages of the persons were as tabulated:

	Community A	Community B	Community C	
Age	16	65	45	
	15	43	30	
	25	77	22	
	30	90	66	
	39	82	47	
	20	69	33	
	16	73	50	
Σx	161	499	293	953.00 (grand total)
x	23	71.29	41.86	45.38 (grand mean)

Is there a significant difference in the ages?

10.4 Measurements on cumulative radiation dosage were made on workers at an atomic weapons plant over a six-month period. The following table presents data for workers whose dosage was assessed at three different locations. Determine whether there was a significant difference in the mean dosage level among the three locations.

	Location A	Location B	Location C	
Cumulative radiation dosage	11	29	37	
	27	41	51	
	19	19	42	
	21	39	28	
	31	24	35	
	14	35	48	
	28	46	75	
	22	64	49	
	18	52	61	
	10	23	52	
Σx	201	372	478	1051 (grand total)
x	20.1	37.2	47.8	35.03 (grand mean)

10.5   a. Describe the differences between a one-way and two-way ANOVA.
   b. What are the assumptions made when one performs an ANOVA?
   c. What H_0 is usually tested with a one-way or a two-way ANOVA?

10.6   a. Why does one use a multiple comparison test such as Tukey's HSD rather than a t test?
   b. How would your results differ if you were to use the t test rather than Tukey's HSD test?

10.7   What are the following critical F values for the $\alpha = .05$ level?
   a. $F_{1,16} = $ _____ ; $F_{3,16} = $ _____ ; $F_{3,36} = $ _____
   b. What are these critical values for $\alpha = .01$?

10.8   a. What are the degrees of freedom for between, within, and total treatments for a one-way ANOVA with 4 treatments and 10 subjects in each treatment?
   b. What are the degrees of freedom for each of the components of a randomized complete block design with 5 treatments and 3 blocks?

10.9   Perform Tukey's HSD test for the following ANOVAs to determine which pairs are significantly different:
   a. Exercise 10.1
   b. Exercise 10.3

10.10   Complete the following ANOVA table:

Source	SS	df	MS	F
Between	360			
Within	450	15		
Total		19		

Is the F ratio significant at the $\alpha = .05$ level?

10.11   a. Prepare an ANOVA table using the equation for unequal sample sizes for the data in Exercise 10.3, assuming that the fourth and fifth observations are missing for Community B and the fourth observation for Community C.
   b. How do the results differ from those when no observations were missing?

10.12   Complete the following ANOVA table:

Source	SS	df	MS	F
Treatment	160	4		
Blocks		5		
Error	200			
Total	600	29		

What is your conclusion regarding the significance of the treatment effect?

10.13   An investigator wants to determine whether there is a significant difference between three different smoking cessation programs in terms of recidivism. He also wants to learn whether being part of a different weight group plays a role in

earlier recidivism. He conjures up a study to see how many days a person was smoke-free during the first 30 days. The following are his data:

Weight Groups	Program A	Program B	C
121–140	30	25	21
141–160	25	23	20
161–180	27	20	22
181–200	25	19	16
201–220	20	18	14
220+	22	14	18

a. Prepare an ANOVA table.
b. Determine whether there is a difference in the three programs.
c. Which program pairs are significant at $\alpha = .05$?
d. Perform a test to determine if weight plays a role in recidivism. Is the MS(blocks) significant at $\alpha = .05$?

10.14 You obtained a calculated F of -4.50. Under what circumstances would you calculate a $-F$ ratio? Explain.

10.15 A researcher wanted to determine if different cereals had varying effects on growth and weight gain. A laboratory experiment was designed so that each of 5 groups of newly weaned rats were fed a diet of a particular brand of cereal. Each group had 7 rats for a total of 35 rats. At the end of the experimental period, the animals were weighed and their weight in ounces recorded in the following table.

	A	B	Brand C	D	E
	9	5	2	6	3
Weight	7	4	1	5	8
gain	8	6	1	5	9
(ounces)	9	5	3	5	2
	6	6	2	6	5
	8	7	2	7	7
	9	2	3	8	1

a. Complete the following ANOVA table.

Source	SS	df	(MS) or s^2	F
Between				
Within		·		
Total				

b. Was the F ratio that was obtained significant? Explain.
c. Perform a Tukey test, if you found a significant F.

11 Inferences Regarding Proportions

Chapter Outline

11.1 Introduction
Discusses the problem of inference in qualitative data

11.2 Mean and Standard Deviation of the Binomial Distribution
Explains how to compute a mean and a standard deviation for the binomial distribution

11.3 Approximation of the Normal to the Binomial Distribution
Shows that, using the normal approximation, it is possible to compute a Z score for a number of successes

11.4 Test of Significance of a Binomial Proportion
Gives instructions on how to test hypotheses regarding proportions if the distribution of the proportion of successes is known

11.5 Test of Significance of the Difference Between Two Proportions
Illustrates that, because the difference between two proportions is approximately normally distributed, a hypothesis test for the difference may be easily set up

11.6 Confidence Intervals
Discusses and illustrates confidence intervals for π and $\pi_1 - \pi_2$

Learning Objectives

After studying this chapter, you should be able to

1. Compute the mean and the standard deviation of a binomial distribution
2. Compute Z scores for specific points on a binomial distribution
3. Perform significance tests of a binomial proportion and of the difference between two binomial proportions
4. Calculate confidence intervals for a binomial proportion and for the difference between two proportions

11.1   INTRODUCTION

Is there a significant difference in the risk of death from leukemia for males and females? Is the proportion of persons who now smoke less than it was at the time of publication of the Surgeon General's Report on the hazards of smoking? These are typical questions that cannot easily be answered by the methods discussed in the previous chapters. Why not? The methods previously discussed are applicable to *quantitative* data such as height, weight, and blood pressure for which a mean and standard error can be computed. The new questions deal with *qualitative* data—data for which individual quantitative measurements are not available, but that relate to the presence or absence of some characteristic, such as smoking. For these data, we have a new statistic, p, the estimate of the true proportion, π, of individuals who possess a certain characteristic. Previously we dealt with $\bar{x}$, the mean value of some characteristic for a group of individuals.

This chapter focuses on (1) the mean and the standard deviation of x, the number of successful events in a binomial experiment, and (2) the mean and the standard error of p, the proportion of successful events observed in a sample. To best understand the difference between the distribution of binomial events (x) and the distribution of the **binomial proportion** (p), try comparing these distributions to those in the approximate analogous quantitative situation. Roughly speaking, the x's of a binomial distribution correspond to the quantitative x's in a distribution with a mean μ and a standard deviation σ. The p's of the binomial correspond to the x's in a distribution with a mean $\mu_{\bar{x}}$ and a standard error $\sigma/\sqrt{n}$.

This chapter considers the tests of significance for proportions, differences between two proportions, and the confidence intervals for both.

11.2   MEAN AND STANDARD DEVIATION OF THE BINOMIAL DISTRIBUTION

In Chapter 5 we learned that the probability of x successful outcomes in n independent trials is given by

$$\binom{n}{x} P^x (1 - P)^{n-x}$$

where P is the probability of a success in one individual trial. To be consistent in using Greek letters to designate unknown parameters, in this chapter we use π to designate the probability of x successful outcomes.

Using mathematical statistics, we can show that in a binomial distribution, the mean for the number of successes, x, is

$$\mu = n\pi \tag{11.1}$$

and the standard deviation is

$$\sigma = \sqrt{n\pi(1 - \pi)} \tag{11.2}$$

1.3 APPROXIMATION OF THE NORMAL TO THE BINOMIAL DISTRIBUTION

The normal distribution is a reasonable approximation to the binomial when n is large. Therefore, we can find the point on the Z distribution that corresponds to a point x on the binomial distribution by using

$$Z = \frac{x - n\pi}{\sqrt{n\pi(1 - \pi)}} \tag{11.3}$$

In Chapter 5 we showed that when the number of trials or cases is greater than 30, it would be quite cumbersome to evaluate the binomial expansion to find the exact probability of the occurrence of a certain event. Mathematical statisticians have demonstrated that the continuous normal distribution is a good approximation to the discrete binomial, providing the following relationships are satisfied:

$$n\pi \geq 5 \quad \text{and} \quad n(1 - \pi) \geq 5$$

Hence, with the use of the well-known equations for the mean and the standard deviation of the binomial distribution, shown in Figure 5.3, it is a simple task to approximate the probability of a binomial event.

■ EXAMPLE 1

A group of physicians treated 25 cases of chronic leukemia, a disease for which the five-year survival rate was known to be .20. They observed that 9 of their patients had survived for five years or more. They wanted to know whether such an event was unusual. What is the probability, out of 25 cases, of observing 9 or more "successes" (i.e., survival for five or more years)?

First, we compute the mean and the standard deviation:

$$\mu = n\pi = (25)(.2) = 5$$
$$\sigma = \sqrt{n\pi(1 - \pi)} = \sqrt{25(.2)(.8)} = 2$$

Then we compute the Z score:
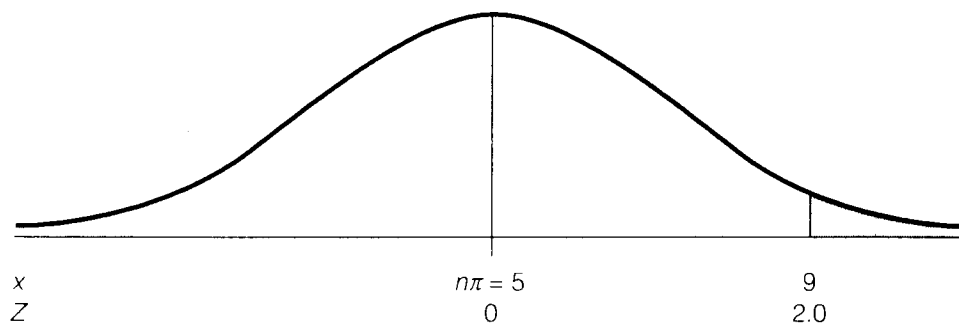
Figure 11.1 Approximation to the Binomial Distribution for Example 1

$$Z = \frac{x - n\pi}{\sqrt{n\pi(1 - \pi)}} = \frac{9 - 5}{\sqrt{25(.2)(.8)}} = \frac{4}{2} = 2.0$$

The result: 9 five-year survivals on the binomial distribution corresponds to a Z of 2.0 on the normal distribution, as shown in Figure 11.1. The area beyond $Z = 2.0$ is .023. Therefore, the probability of five-year survival for at least 9 of 25 patients is .023, whereas the probability of five-year survival for 1 patient is .20.

∎

Because we are using a normal (continuous) distribution to approximate a discrete one, we may apply the **continuity correction** to achieve an adjustment. This correction is made by subtracting one-half from the absolute value of the numerator; that is,

$$Z = \frac{|x - n\pi| - 1/2}{\sqrt{n\pi(1 - \pi)}}$$

$$= \frac{|9 - 5| - 1/2}{\sqrt{25(.2)(.8)}} = \frac{3.5}{2} = 1.75$$

and $P(Z > 1.75) = .0401$, a result nearly two times that obtained without the correction. The continuity correction will not make a large difference when n is large.

When n is very large and π is very small, another important distribution, the **Poisson distribution,** is a good approximation to the binomial. It deals with discrete events that occur infrequently. For a treatment of this subject, see more advanced textbooks, such as Armitage (1971).

11.4   TEST OF SIGNIFICANCE OF A BINOMIAL PROPORTION

The previous section considered the distribution of the binomial event x. This section considers the distribution of the binomial proportion p, which is similar to considering the distribution of $\bar{x}$ for quantitative data.

The mean of the distribution of a binomial proportion p is given by the population parameter

$$\pi = \frac{x}{n} = \frac{\text{number of successes in the population}}{\text{number of cases in the population}} \qquad (11.4)$$

and the standard error of p is given by

$$\sigma_p = \sqrt{\frac{\pi(1 - \pi)}{n}} \qquad (11.5)$$

Because p appears to be normally distributed, providing n is reasonably large, we can find the Z score corresponding to a particular p and perform a test of significance.

■ **EXAMPLE 2**

There were 245 deaths from leukemia in California one year. Of these, 145 were males,

$$p = \frac{145}{245} = .59$$

and 100 were females,

$$1 - p = \frac{100}{245} = .41$$

Is .59, the observed proportion of male deaths, significantly different from the expected .49, the proportion of males in the California population?

$$\pi = .49 \qquad 1 - \pi = .51 \qquad n = 245$$

$$SE(p) = \sqrt{\frac{\pi(1 - \pi)}{n}} = \sqrt{\frac{(.49)(.51)}{245}} = .032$$

Using the steps of a test of a hypothesis, we get the following results:

1. H_0: $\pi = .49$; there is no sex difference in the proportion of deaths.
2. $\alpha = .05$.
3. Test statistic:

$$Z = \frac{p - \pi}{SE(p)} = \frac{.59 - .49}{.032} = \frac{.10}{.032} = 3.12 \qquad (11.6)$$
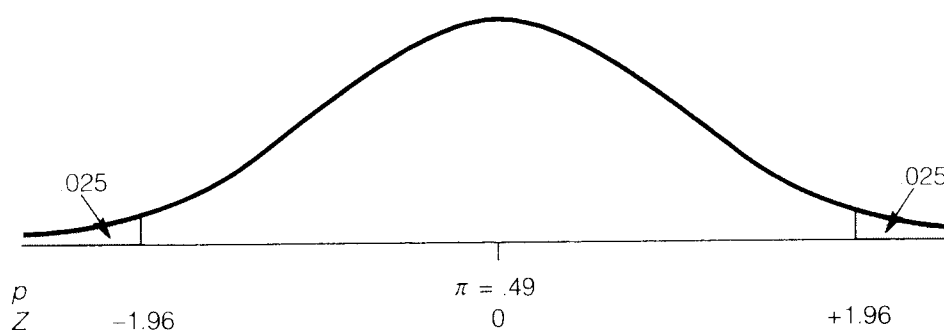
Figure 11.2 Critical Region for Example 2

4. Critical region: From the Z distribution (Table A, back inside cover), we find that Z is ± 1.96 (Figure 11.2).

5. The computed Z of 3.12 is greater than the critical value of 1.96, so we reject the null hypothesis that the proportion of deaths from leukemia is the same for both sexes and conclude that the risk of dying from this disease is greater for males than for females. If we apply the continuity correction in this example, we will have

$$Z = \frac{|.59 - .49| - 1/2n}{.032}$$

$$= \frac{.10 - .002}{.032} = \frac{.098}{.032} = 3.06$$

which actually makes very little difference in the result. ∎

11.5 TEST OF SIGNIFICANCE OF THE DIFFERENCE BETWEEN TWO PROPORTIONS

In practice, you seldom have a convenient population proportion for comparison. More commonly, you will be called upon to compare proportions from two different samples, possibly one from a control group and the other from a treatment group; that is, we assume that $\pi_1 = \pi_2$ in estimating $SE(p_1 - p_2)$. In such a case, you want to learn if p_1, the proportion with the given characteristic in one sample, differs significantly from p_2, the proportion with the same characteristic in the other sample. To do this, you need to know the distribution of the differences $(p_1 - p_2)$ and the mean and the standard error of this distribution. Mathematical statisticians have shown that $p_1 - p_2$ follows a nearly normal distribution. The mean is

$$\mu = p_1 - p_2 \tag{11.7}$$

The standard error is estimated by

$$SE(p_1 - p_2) = \sqrt{\frac{p'q'}{n_1} + \frac{p'q'}{n_2}} \qquad (11.8)$$

where

$$p' = \frac{x_1 + x_2}{n_1 + n_2} \quad \text{and} \quad q' = 1 - p' \qquad (11.9)$$

and

$$p_1 = \frac{x_1}{n_1} \qquad (11.10)$$

and

$$p_2 = \frac{x_2}{n_2} \qquad (11.11)$$

Knowing the mean and the standard error of the distribution of differences, we can calculate a Z score:

$$Z = \frac{p_1 - p_2 - (\pi_1 - \pi_2)}{SE(p_1 - p_2)} \qquad (11.12)$$

If $\pi_1 \neq \pi_2$, the formula for $SE(p_1 - p_2)$ is

$$\sqrt{\frac{\pi_1(1 - \pi_1)}{n_1} + \frac{\pi_2(1 - \pi_2)}{n_2}}$$

■ **EXAMPLE 3**

A public health official wishes to know how effective health education efforts are regarding smoking. Of 100 males sampled in 1965 at the time of release of the Surgeon General's Report on the Health Consequences of Smoking, 51 were found to be smokers. In 1990, a second random sample of 100 males, similarly gathered, indicated that 31 were smokers. Is the reduction in proportion from .51 to .31 statistically significant?

$$p_1 = \frac{51}{100} = .51 \qquad p_2 = \frac{31}{100} = .31$$

$$p' = \frac{51 + 31}{100 + 100} = \frac{82}{200} = .41$$

$$SE(p_1 - p_2) = \sqrt{\frac{(.41)(.59)}{100} + \frac{(.41)(.59)}{100}}$$

$$= \sqrt{.004838} = .070$$

Again we apply the steps for a hypothesis test:

1. H_0: $\pi_1 - \pi_2 \leq 0$ (there has not been a reduction in smoking) versus H_1: $\pi_1 - \pi_2 > 0$ (there has been a reduction).

2. $\alpha = .05$.

3. Test statistic:

$$Z = \frac{p_1 - p_2 - 0}{SE(p_1 - p_2)} = \frac{.51 - .31}{.07} = \frac{.20}{.07} = 2.86$$

4. Critical region: The Z distribution (Table A) shows $Z > 1.64$.

5. The computed Z of 2.86 is more than the critical value of 1.64. Consequently, on the basis of the information of this sample, the official rejects the null hypothesis that there has not been a significant reduction in cigarette smoking 25 years after publication of the Surgeon General's Report. ■

11.6   CONFIDENCE INTERVALS

Although hypothesis testing is useful, we often need to go another step to learn, say, the true proportion of male smokers in 1990 or the true difference in the proportion of male smokers between 1990 and 1965. To deal with such questions, we compute confidence intervals for π and for $\pi_1 - \pi_2$ by employing a method parallel to the one used for computing confidence intervals for μ and $\mu_1 - \mu_2$.

Confidence Interval for π

In Chapter 8 we found the confidence interval of μ to be

$$x \pm Z \frac{\sigma}{\sqrt{n}}$$

Similarly, the confidence interval for π is

$$p \pm Z \sqrt{\frac{\pi(1 - \pi)}{n}}$$

This expression presents a dilemma: It requires that we know π, which is unknown. The way out of this puzzle is to have a sufficiently large sample size, permitting the use of p as an estimate of π. The expression then becomes

$$p \pm Z \sqrt{\frac{p(1-p)}{n}} \tag{11.13}$$

The solution for small sample sizes is known but is derived from the binomial distribution. Some statistical textbooks feature them.

■ EXAMPLE 4

In the previous example the public health official estimated that the proportion of male smokers in 1990 was .31. As this was only a sample estimate, the official also needed to obtain a confidence interval to bracket the true π and therefore calculated as follows:

$$95\% \text{ CI for } \pi = p \pm 1.96 \sqrt{\frac{p(1-p)}{n}}$$

$$= .31 + 1.96 \sqrt{\frac{(.31)(.69)}{100}}$$

$$= .31 \pm .09$$

$$= (.22, .40)$$

The official now could have 95% confidence that the true proportion of male smokers in 1990 was between .22 and .40. ■

Confidence Interval for the Difference of $\pi_1 - \pi_2$

The confidence interval for the difference of two means is

$$\text{CI for } \mu_1 - \mu_2 = \bar{x}_1 - \bar{x}_2 \pm Z[\text{SE}(\bar{x}_1 - \bar{x}_2)]$$

The confidence interval for the difference of two proportions is similar:

$$\text{CI for } \pi_1 - \pi_2 = p_1 - p_2 \pm Z \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} \tag{11.14}$$

■ EXAMPLE 5

To find the confidence interval on the true difference between male smokers in 1990 and 1965, the public health official would perform the following calculation:

$$95\% \text{ CI for } \pi_1 - \pi_2 = p_1 - p_2 \pm 1.96 \sqrt{\frac{p_1(1 - p_1)}{n_1} + \frac{p_2(1 - p_2)}{n_2}}$$

$$= .51 - .31 \pm 1.96 \sqrt{\frac{.51(.49)}{100} + \frac{.31(.69)}{100}}$$

$$= .20 \pm .133$$

$$= (.067, .333)$$

These figures would give the official 95% confidence that the reduction in percentage of smokers may have ranged from 6.7% to 33.3% over the 25-year period. ■

Conclusion

The normal approximation to the binomial is a useful statistical tool. It helps answer questions regarding qualitative data involving proportions where individuals are classified into two categories. The mean and the standard deviation are, respectively, $\mu = n\pi$ and $\sigma = \sqrt{n\pi(1 - \pi)}$, giving a Z score of $(x - n\pi)/\sqrt{n\pi(1 - \pi)}$. With an understanding of the distribution of the binomial proportion p and of the distribution of the difference between two proportions, $p_1 - p_2$, we can perform tests of significance and calculate confidence intervals.

Vocabulary List

binomial proportion   continuity correction   Poisson distribution

Exercises

11.1   For the Honolulu Heart Study data of Table 3.1, compute
    a. the proportion of individuals in each education category
    b. the proportion of smokers and nonsmokers
    c. the proportion for each physical activity level

11.2   Using your results from Exercise 11.1b, calculate estimates of the mean and the standard deviation of the proportion of smokers.

11.3   Given that the proportion of smokers in the United States is .31, test to see if the proportion of smokers in Honolulu is significantly different from the national proportion. Use $\alpha = .05$.

11.4   What is the 95% confidence interval for the proportion of smokers in Honolulu for 1969? Refer to Exercise 11.1b.

11.5   In a study of hypertension and taste acuity, one variable of interest was smoking status. Of the 7 persons in the hypertensive group, 4 were smokers. The control group of 21 normotensive persons included 7 smokers. Is there a difference in the proportion of smokers in the two groups at the .05 level of significance?

11.6 Construct a 90% confidence interval for the difference in the proportions of smokers in the hypertensive and normotensive groups of Exercise 11.5.

11.7 In a study of longevity in a village in Ecuador, 29 persons in a population of 99 were age 65 or older. If it is also known that 20% of the U.S. population is 65 or over, does it appear that the proportion of Ecuadorian villagers surviving to 65 and beyond exceeds that of people in the United States? Use $\alpha = .01$.

11.8 Calculate a 99% confidence interval for the proportion of Ecuadorians (Exercise 11.7) who are age 65 or over.

11.9 Of 186 participants in a program to control heart disease, it was discovered that 102 had education beyond secondary school. Does this indicate that the program is attracting a more highly educated group of people than would be expected, given that 25% of the U.S. population has education beyond secondary school? Use $\alpha = .01$.

11.10 In a study of drug abuse among adults, 55 of 219 "abusers" and 117 of 822 "nonusers" stated they started smoking cigarettes at age 12 or younger. Do these data indicate there is a significant difference in the proportions of abusers and nonusers who took up smoking at an early age?

11.11 In a dental study of a tie between infant occlusion and feeding methods, there were 27 breast-fed and 60 bottle-fed infants. It was noted that 7 of the breast-fed babies and 26 of the bottle-fed babies developed a related open-bite gum pad in the first four months of life. Would you conclude that the bottle-fed group showed a higher proportion of the open-bite gum pad problem? Use $\alpha = .05$.

11.12 Compute the following confidence intervals for the difference in proportions, $\pi_1 - \pi_2$:
a. 99% CI for Exercise 11.10
b. 95% CI for Exercise 11.11

11.13 a. What are the mean and standard deviation of x, the number of successes in a binomial distribution?
b. What is the difference between p and π?
c. What are the mean and standard deviation for the binomial proportion p?
d. Under what condition is the normal distribution a reasonable approximation to the binomial distribution?

11.14 Public health officials found that, in a random sample of 100 men in a small community, 13 were infected with AIDS.
a. Obtain an estimate of the proportion of men infected with AIDS in that community.
b. Calculate the 95% CI for π, the true proportion of men infected with AIDS.

11.15 A random check of drivers on a busy highway revealed that 60 out of 100 male drivers and 70 out of 100 female drivers were wearing their seat belts.
a. Obtain estimates of the proportion of male and female drivers who wear seat belts.
b. Construct a 99% CI for $\pi_1 - \pi_2$, the true difference of wearing seat belts between males and females.
c. Is the observed difference between males and females significant at the $\alpha = .01$ level?

11.16 A survey of 100 women and 100 men indicated that 49 of the women and 35 of the men said they are trying to lose weight.
- a. Estimate the difference in the proportion desiring to lose weight between men and women.
- b. Perform a test of significance to determine whether or not this difference is significant at the $\alpha = .05$ level.
- c. Calculate a 95% CI for $\pi_1 - \pi_2$.
- d. Do the results from (b) and (c) support or contradict each other? Why?

11.17 A nationwide survey of medical complaints indicated that 43 out of 100 people in the Southwest and 22 out of 100 people in the region close to the nation's capital suffered from allergies. Is this a chance difference? Are the data consistent with the hypothesis that geography plays a role? (Use $\alpha = .01$.)

11.18 A fitness survey found that 35 out of 100 women and 25 out of 100 men did not exercise. Is this likely to be a real difference or can it be explained by chance? Construct a 95% CI for the difference and state your conclusion.

11.19 A random sample of 100 industrial workers found that 13 of them were exposed to toxic chemicals routinely on their job. Prepare a report that will provide management with information regarding the magnitude of this problem. What statistic or statistics would you include in your report?

11.20 As of September, 1996, 14 states had lowered the legal blood alcohol limit from 0.10% to 0.08%. A "study was undertaken to a__ess whether, relative to nearby states, states adopting a 0.08% legal limit experienced a reduction in the proportion of fatal crashes involving (1) fatally injured drivers with blood alcohol levels of 0.08% or higher and 0.15% or higher, and (2) any driver with a blood alcohol level of 0.08% or higher and 0.15% or higher." Two comparison states were Oregon (0.08%) and Washington.

	Before 0.08% Law		After 0.08% Law	
	Fatally Injured Drivers	Drivers at 0.08% or Higher	Fatally Injured Drivers	Drivers at 0.08% or Higher
Oregon (0.08%)	1275	4455	1023	4186
Washington	1735	6184	1582	5390

NOTE: These data were extrapolated and based on the study by Hingson, Hereen, and Winter (1996).

- a. First, calculate a proportion for each state, before and after the 0.08% law went into effect. You will calculate a total of 4 proportions.
- b. Following the procedure explained in Section 11.5, including Example 3, calculate a test of significance for the difference before and after the new law for Oregon and then do the same for Washington.
- c. Are your results significant for either state? Explain the importance of your findings. Remember that Oregon changed to 0.08% and Washington did not.

12 The Chi-Square Test

Chapter Outline

Learning Objectives

After studying this chapter, you should be able to

1. Indicate the kinds of data and circumstances that call for a chi-square test
2. Compute the expected value for a chi-square contingency table
3. Compute a chi-square statistic and its appropriate degrees of freedom
4. Explain the meaning of degrees of freedom
5. Indicate the type of hypothesis that can be tested with chi-square
6. Find the critical region for a chi-square test
7. Compute two different measures of the strength of association of factors reported in 2×2 tables

12.1   RATIONALE FOR THE CHI-SQUARE TEST

Although the t test is popular and widely used, it may not be appropriate for certain health science problems that call for tests of significance. Because the t test requires data that are quantitative, it is simply not applicable to qualitative data. In other chapters, whenever means or standard deviations were computed, we worked with measurement data. With such data, we were able to record a specific value for each observation. These represented quantitative variables such as height, weight, and cholesterol level. But we are often obliged to classify persons into such categories as male or female, hypertensive or normotensive, and smoker or nonsmoker, and to count the number of observations falling in each category. The result is **frequency data.** In addition, we often have to deal with **enumeration data,** because we enumerate the number of persons in each category; **categorical data,** because we count the number of persons falling into each category; and, as mentioned earlier, **qualitative data,** because we group the categories according to some quality of interest.

Categorical data are not used to quantify blood pressure levels, for example, but rather to classify persons as hypertensive or normotensive. The classification table used to do this is called a **contingency table.** Its use, though, does not permit us to determine whether there is a relationship between two variables by means of a correlation coefficient, because we do not have quantitative x and y observations for each person. Instead, we could perform a **chi-square test** to determine whether there is some association between the two variables. This chapter considers various chi-square tests to deal with such a case and related ones for frequency data.

12.2   THE BASICS OF A CHI-SQUARE TEST

For a given phenomenon, the chi-square test compares the **observed frequencies** with the **expected frequencies.** The expected frequency is calculated from

some hypothesis. To illustrate, let us take the simple example of trying to determine whether a coin is fair.

Suppose you toss a coin 100 times and you observe that heads (H) come up 40 times and tails (T) 60 times. If you hypothesize that the coin is fair, you would expect heads and tails to occur equally—that is, 50 times each. In comparing the observed frequency (O) with the expected frequency (E), you need to determine whether the *deviations* ($O - E$) are significant. As you can see in Table 12.1, if you were to sum the deviations, the total would equal zero, as indicated in column 3.

To avoid this problem, you might first square each deviation, as in column 4. This approach has a problem, too: The same value is obtained for equal deviations regardless of magnitude. For instance, consider $O - E$ for two possibilities: $60 - 50 = 10$ and $510 - 500 = 10$. Arithmetically, the deviations are identical, but they are far from identical in meaning; although a deviation of 10 from an expected 50 is impressive, the same deviation from an expected 500 is hardly noticeable. The best way of overcoming this problem is to look at the proportional squared deviations, $(O - E)^2/E$. Here, the two possibilities become $(60 - 50)^2/50 = 2.0$ and $(510 - 500)^2/500 = .02$. Now the deviations offer a more meaningful statistical perspective. From column 5 of Table 12.1, we can see that for the coin problem, the sum of the proportional squared deviations is equal to 4.

The next question is whether the value we have just calculated,

$$\sum \frac{(O - E)^2}{E} = 4$$

can occur easily by chance or whether it is an unusual event that is unlikely to occur by chance except in rare instances, say less than 5% of the time. To resolve this question, we need to know how the quantity, designated as χ^2 (chi-square), is distributed; that is, we have to determine the probability distribution for the statistic

$$\chi^2 = \sum \frac{(O - E)^2}{E} \tag{12.1}$$

Mathematical statisticians have shown that this quantity is approximated quite well by the **chi-square distribution** if the sample sizes and the expected numbers are not too small. This distribution is positively skewed, beginning at zero. By figuring out the area beyond 4 on a chi-square distribution, we can determine a p value and either accept or reject the hypothesis.

There is, in fact, a family of chi-square distributions. The correct one to use depends, as in the t distribution, on a quantity called the degrees of freedom. For chi-square, degrees of freedom are determined as the number of *independent* deviations (each $O - E$) in the contingency table. A two-cell table (e.g., Table 12.1) has 1 df. Wherever you can determine expected frequencies from your

Table 12. 1 Observed and Expected Frequencies and Their Deviations for 100 Tosses of a Coin

		(1)	(2)	(3)	(4)	(5)
		O	E	$O - E$	$(O - E)^2$	$\dfrac{(O - E)^2}{E}$
H		40	50	-10	100	2
T		60	50	10	100	2
	Total	100	100	0	200	4

hypothesis, the degrees of freedom are one less than the number of categories. The coin problem has two categories, heads and tails, so there is 1 df. If you were trying to determine whether a six-sided die was unbiased, you would have $6 - 1 = 5$ df.

In Figure 12.1 you can see the shapes of several chi-square distributions. For each, the upper 5% of the area is shaded. Note that as the degrees of freedom increase, so does the critical value needed to reject a null hypothesis. Intuitively, this sounds right: Because the degrees of freedom are proportional to the number of independent categories, you would well expect the critical chi-square value to increase with more categories.
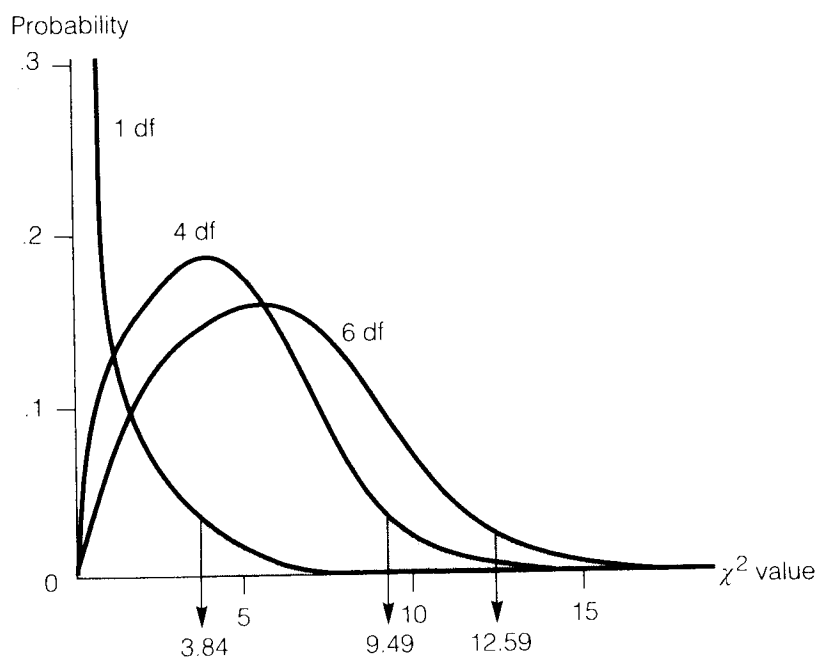
Figure 12.1   The Chi-Square Distribution for Varying Degrees of Freedom

Table 12.2 The Probability of Exceeding the Chi-Square Value in the Chi-Square Distribution

df	α							
	.99	.95	.90	.50	.10	.05	.01	.001
1	.00157	.00393	.0158	.455	2.706	3.841	6.635	10.827
2	.0201	.103	.211	1.386	4.605	5.991	9.210	13.815
3	.115	.352	.584	2.366	6.251	7.815	11.345	16.226
4	.297	.711	1.064	3.357	7.779	9.488	13.277	18.467
5	.554	1.145	1.610	4.351	9.236	11.070	15.806	20.515
6	.872	1.635	2.204	5.348	10.645	12.592	16.812	22.457
7	1.239	2.167	2.833	6.346	12.017	14.067	18.475	24.322
8	1.646	2.733	3.490	7.344	13.362	15.507	20.090	26.125
9	2.088	3.325	4.168	8.343	14.684	16.919	21.666	27.877
10	2.558	3.940	4.865	9.342	15.987	18.307	23.209	29.588
11	3.053	4.575	5.578	10.341	17.275	19.675	24.725	31.264
12	3.571	5.226	6.304	11.340	18.549	21.026	26.217	32.909
13	4.107	5.892	7.042	12.340	19.812	22.362	27.688	34.528
14	4.660	6.571	7.790	13.339	21.064	23.685	29.141	36.123
15	5.229	7.261	8.547	14.339	22.307	24.996	30.578	37.697
20	8.260	10.581	12.443	19.337	28.412	31.410	37.566	43.315
30	14.953	18.493	20.599	29.336	40.256	43.773	50.892	59.703
40	22.164	26.509	29.051	39.335	51.805	55.759	63.691	73.402
50	29.707	34.764	37.689	49.335	63.167	67.505	76.154	86.661
60	37.485	43.188	46.459	59.335	74.397	79.082	88.379	99.607

Table 12.2 gives the critical values for the chi-square distribution for various degrees of freedom. Here you can see that the upper 5% chi-square value for 1 df is 3.84, for 4 df is 9.49, and for 6 df is 12.59.

Back to our original question: "Is the coin fair?" Recall that the χ^2 sum was 4. For 1 df, this falls within the upper 5% critical region. Therefore, you would reject the H_0 that the coin is fair; that is, you would not expect to observe a deviation as large as (or larger than) this to occur by chance alone. Your conclusion: The coin is probably unbalanced or loaded or was not properly thrown. A point to note is that the chi-square test, unlike some others, is a one-tailed test. The rationale for this is that we are almost always concerned only about whether the deviations are too large, seldom about whether they are too small. For example, we would worry about a dangerously high level of air pollution, but certainly not about too low a level.

.3   TYPES OF CHI-SQUARE TESTS

In practical applications, you will often encounter problems involving two variables. Specifically, you may employ chi-square tests to determine

1. Independence (if any) between the two variables
2. Whether various subgroups are homogeneous
3. Whether there is a significant difference in the proportions in the sub-classes among the subgroups

We will discuss each of these tests.

12.4   TEST OF INDEPENDENCE BETWEEN TWO VARIABLES

Kuzma and Kissinger (1981) published a study of the effects that maternal use of alcohol during pregnancy have on the newborn. Some of their data, regarding smoking and drinking, are shown in Table 12.3. Here you can see that 30.5% of the nondrinking women and 67.3% of the heaviest drinkers smoked during their pregnancies. We might wonder whether drinking and smoking are dependent variables and whether the relationship is explainable by chance. A way to approach this question is to test the null hypothesis that there is no relationship between smoking and drinking during pregnancy. To do this, we need to know the expected values before we can compute a χ^2 statistic. Expected values can be generated from the null hypothesis, which states there is no relationship between drinking and smoking during pregnancy.

For purposes of this discussion, we set up a special notation, in which the eight cells of Table 12.3 are identified as $E_{11}, \ldots, E_{24}$, as shown in Table 12.4. The

Table 12.3 Number and Percentages (in parentheses) of 11,127 Pregnant Women by Alcohol and Drinking Status

Smoking Status	Alcohol Consumption				
	None	Low	Medium	High	Total
Smokers	1,880 (30.5%)	2,048 (45.7%)	194 (53.0%)	76 (67.3%)	4,198 (37.7%)
Nonsmokers	4,290 (69.5%)	2,430 (54.3%)	172 (47.0%)	37 (32.7%)	6,929 (62.3%)
Total	6,170 (55.5%)	4,478 (40.2%)	366 (3.3%)	113(1.0%)	11,127(100.0%)

Table 12.4 Notation for Expected Frequencies of a Two-Variable Table

Smoking Status	Alcohol Consumption				
	None	Low	Medium	High	Total
Smokers	E_{11}	E_{12}	E_{13}	E_{14}	T_s
Nonsmokers	E_{21}	E_{22}	E_{23}	E_{24}	T_{ns}
Total	T_{nd}	T_{ld}	T_{md}	T_{hd}	T

probability multiplication rule states that the probability of two independent events A and B is $P(A \text{ and } B) = P(A)P(B)$.

We are testing the hypothesis that the two variables are independent. Therefore we can apply the multiplication rule to obtain the frequencies expected if the hypothesis of independence is indeed true; that is, from the data in Table 12.4, the probability of a woman's being in the smoking group (A) and in the nondrinking group (B) is

$$P(A)P(B) = \left(\frac{4198}{11,127}\right)\left(\frac{6170}{11,127}\right) = (.377)(.555) = .2092$$

$$= \left(\frac{T_s}{T}\right)\left(\frac{T_{nd}}{T}\right)$$

where T_s = total smokers and T_{nd} = total nondrinkers. Therefore the expected number of smokers who are also nondrinkers is

$$E_{11} = 11,127(.2092) = 2327.8$$

The meaning of E_{11} is what you would expect, assuming the null hypothesis to be true—that 2328 of the smokers will be nondrinkers. Continuing in the same way, we can obtain expected frequencies for all cells: for low, medium, and high alcohol consumption and for the nonsmoking categories. Thus

$$E_{12} = (.37728)(.40244)(11,127) = 1689.4$$

$$E_{13} = (.37728)(.03289)(11,127) = 138.1$$

$$E_{24} = (.62272)(.010155)(11,127) = 70.4$$

Although it may seem absurd to compute expected values to a fraction of a person, this is often done in order to avoid roundoff error and ensure that "expected" and "observed" row totals are identical. All expected frequencies are shown in Table 12.5. Now we can proceed to compute the χ^2 statistic:

Table 12.5 Observed and Expected Frequency of Alcohol Consumption and Smoking During Pregnancy for 11,127 Women

	Alcohol Consumption							
	None		Low		Medium		High	
Smoking Status	O	E	O	E	O	E	O	E
Smokers	1880	2327.8	2048	1689.4	194	138.1	76	42.7
Nonsmokers	4290	3842.2	2430	2788.5	172	227.9	37	70.4
Total	6170		4478		366		113	

$$\chi^2 = \sum \frac{(O-E)^2}{E} = \frac{(1880 - 2327.8)^2}{2327.8} + \frac{(2048 - 1689.4)^2}{1689.4}$$

$$+ \frac{(194 - 138.1)^2}{138.1} + \frac{(76 - 42.7)^2}{42.7} + \frac{(4290 - 3842.2)^2}{3842.2}$$

$$+ \frac{(2430 - 2788.5)^2}{2788.5} + \frac{(172 - 227.9)^2}{227.9} + \frac{(37 - 70.4)^2}{70.4} = 338.7$$

Is a χ^2 of 338.7 significant? To find out, we check Table 12.2 for the critical value. But first, we need to know the number of degrees of freedom. In the case of our example, where we do not know the expected frequencies **a priori** (i.e., by deductive reasoning) but have obtained them from the data, the degrees of freedom are equal to $(c - 1)(r - 1)$, where c is the number of columns and r the number of rows. Here we have 4 columns and 2 rows; therefore, df = $(4 - 1)(2 - 1) = 3$.

From Table 12.2 we find the critical 5% value for 3 df to be 7.8. Because the computed χ^2 of 338.7 falls well into the critical region, we reject the hypothesis of independence between drinking and smoking during pregnancy. This suggests that there is an association between smoking and drinking among pregnant women.

The preceding discussion should help you understand the meaning of degrees of freedom. Please note that the expected values for each category add up to the total observed value for that category. Note also that we could have computed expected values for only three of the eight cells, with the others obtained by subtraction. These three cells represent the three "independent" quantities—that is, the 3 df. The other five quantities are not "independent" because they can be obtained by subtracting the first three from column or row totals.

12.5   TEST OF HOMOGENEITY

It is often important to determine whether the distribution of a particular characteristic is similar for various groups. To do this, we can perform a chi-square test called a test of homogeneity.

■ EXAMPLE 1

From the alcohol–pregnancy study of Kuzma and Kissinger (1981), we have data on the distribution of drinkers by ethnic group. As shown in Table 12.6, among Caucasians, 51.2% were abstainers, 43.6% light drinkers, 3.9% medium drinkers, and 1.2% heavy drinkers. The percentage distribution is fairly similar among the ethnic groups, except that the Caucasian group includes fewer abstainers and more drinkers in all categories. Is this difference real or no greater than would be expected by chance? That is, can we assume that groups of preg-

Table 12.6 Drinking Status During Pregnancy, by Ethnic Group

	Alcohol Consumption									
	None		Light (<1.0 oz*)		Medium (1.0–2.99 oz)		Heavy (≥3.00 oz)		Total	
Ethnicity	n	%	n	%	n	%	n	%	n	%
Black	411	60.4	253	37.2	12	1.8	5	0.7	681	6.3
Hispanic	1,459	64.0	757	33.2	53	2.3	10	0.4	2,279	21.2
Caucasian	3,732	51.2	3,179	43.6	284	3.9	90	1.2	7,285	67.7
Other	322	61.6	187	35.8	10	1.9	4	0.8	523	4.9
Total	5,924	55.0	4,376	40.6	359	3.3	109	1.0	10,768	100.0

*Equivalent ounces of absolute alcohol per day.

nant women of various ethnicity tend to have essentially the same drinking patterns?

To test for homogeneity, we again need to establish the expected frequencies, this time basing them on a somewhat different rationale than the probability argument used in Section 12.4. Nevertheless, the equations used to obtain expected frequencies are the same. For example, the expected number of abstainers among Caucasian women is computed as

$$E_{13} = \left(\frac{5924}{10,768}\right)\left(\frac{7285}{10,768}\right)(10,768) = 4007.8$$

The other expected frequencies are obtained similarly and are shown in parentheses in Table 12.7. Having the expected frequencies, we can now proceed with the test of significance as follows:

1. H_0: The several ethnic groups are homogeneous in their drinking patterns.
   H_1: The several groups are not homogeneous in their drinking patterns.

Table 12.7 Observed and Expected Frequencies of Alcohol Intake During Entire Pregnancy, by Ethnic Group

	Alcohol Consumption								
	None		Light		Medium		Heavy		
Ethnicity	O	E	O	E	O	E	O	E	Total
Black	411	(374.7)	253	(276.8)	12	(22.7)	5	(6.9)	681
Hispanic	1,459	(1,253.8)	757	(926.2)	53	(76.0)	10	(23.1)	2,279
Caucasian	3,732	(4,007.8)	3,179	(2,960.5)	284	(242.9)	90	(73.7)	7,285
Other	322	(287.7)	187	(212.5)	10	(17.4)	4	(5.3)	523
Total	5,924		4,376		359		109		10,768

Figure 12.2 Critical Region for χ^2_9

2. $\alpha = .05$.

3. Critical region: The critical region for χ^2 with $(c - 1)(r - 1) = (4 - 1)(4 - 1) = 9$ df (denoted as χ^2_9) is shown in Figure 12.2 to be 16.9.

4. Test statistic:

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

$$= \frac{(411 - 374.7)^2}{374.7} + \frac{(253 - 276.8)^2}{276.8} + \cdots + \frac{(4 - 5.3)^2}{5.3}$$

$$= 146.3$$

5. The computed χ^2 of 146.3 falls in the critical region, so we conclude that the deviations in drinking patterns among the various ethnic groups are not homogeneous; that is, the various ethnic groups do not appear to be homogeneous in their drinking patterns. ∎

12.6   TEST OF SIGNIFICANCE OF THE DIFFERENCE BETWEEN TWO PROPORTIONS

Another application of the chi-square test is in learning whether the proportion of successes in a treated group differs significantly from the proportion in a control group. It can be considered an alternative to the Z test for a 2×2 table.

■ **EXAMPLE 2**

For some years there has been a lively medical controversy over the efficacy of vitamin C in preventing the common cold. Several studies concluded that vitamin C was no more effective than a placebo. In Table 12.8, which presents some unpublished data from one such study, we find that 63% of the children treated

Table 12.8 Number and Frequencies of Children Developing Colds, by Vitamin C and Placebo Groups

Status	Vitamin C Group		Placebo Group		Total
Children free of colds	21	(37%)	11	(24%)	32
Children developing colds	36	(63%)	35	(76%)	71
Total	57	(100%)	46	(100%)	$n = 103$

with vitamin C and 76% of the placebo group caught colds. Does the number developing colds differ between the two groups?

The expected frequencies for Table 12.8 are

$$E_{11} = \frac{(32)(57)}{103} = 17.7$$

By subtraction, the remaining expected frequencies are $E_{12} = 14.3$, $E_{21} = 39.3$, and $E_{22} = 31.7$. The value of the test statistic is

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

$$= \frac{(21 - 17.7)^2}{17.7} + \frac{(11 - 14.3)^2}{14.3} + \frac{(36 - 39.3)^2}{39.3} + \frac{(35 - 31.7)^2}{31.7}$$

$$= .61 + .76 + .28 + .34$$

$$= 1.99$$

As before, there are $(c - 1)(r - 1)$ df. In this example, $(c - 1)(r - 1) = 1$. The critical χ^2 at the 5% level for 1 df is 3.84.

The resulting χ^2 of 1.99 is not within the critical region; therefore we fail to reject the hypothesis that the percentage with colds in both groups is the same. So we could logically conclude that, for this size sample, the observed difference of children free of colds between 37% − 24% = 13% could well have occurred by chance. ■

12.7 TWO-BY-TWO CONTINGENCY TABLES

Perhaps the most common chi-square analysis used in health research involves data presented in a 2 × 2 (fourfold) table in which there are two groups and two possible responses. Table 12.9 is a generalized representation of such a table. The observed frequencies are represented symbolically by the letters a, b, c,

Table 12.9 Schematic Representation for 2 × 2 Contingency Table

Response	Treatment	Control	Total
Yes	a	b	$a + b$
No	c	d	$c + d$
	$a + c$	$b + d$	$a + b + c + d = n$

and d. With such data, it is possible to compute the χ^2 statistic directly, avoiding the need to compute expected frequencies:

$$\chi^2 = \frac{n(ad - bc)^2}{(a + c)(b + d)(a + b)(c + d)} \tag{12.2}$$

Thus, using the data on vitamin C in Example 2, we obtain the same result as in that example:

$$\chi^2 = \frac{103[(21)(35) - (11)(36)]^2}{(57)(46)(32)(71)} = 1.99$$

The equations we use to compute χ^2 result in approximations to the chi-square distribution. They are quite close for many degrees of freedom, not too close for a few, and not as good for 1 df. Just as we always use discrete observations to approximate a statistic that is continuously distributed, it is desirable to apply a correction for this. A frequently used solution is the **Yates continuity correction** for chi-squares with 1 df. However, Grizzle (1967) has shown that, because the correction is too conservative in that it leads too often to nonrejection of the null hypothesis, many practicing statisticians do not recommend its use.

■ EXAMPLE 3

A survey on the use of seat belts found that 24 out of 60 males with a high school education and 30 out of 40 college graduates wore seat belts regularly. Is there evidence suggesting an association between education and seat belt use? Table 12.10 presents the data in a 2 × 2 contingency table.

Using equation 12.2 to compute the χ^2, we have

$$\chi^2 = \frac{n(ad - bc)^2}{(a + c)(b + d)(a + b)(c + d)}$$

$$= \frac{100[(24)(10) - (36)(30)]^2}{(54)(46)(60)(40)}$$

$$= 11.8$$

Table 12.10 A 2 × 2 Contingency Table of Seat Belt Use and Education of a Sample of 100 Men

Education	Used Seat Belt		
	Yes	No	Total
High School Graduate	24	36	60
College Graduate	30	10	40
Total	54	46	100

Because the computed χ^2 of 11.8 is larger than the critical χ^2 of 3.84, with 1 df, we would reject the H_0 of independence; that is, we would suspect that there is an association between education and seat belt use. ■

For those who insist on the use of the Yates correction, which was proposed by Yates in 1934, to subtract one-half of the total number of observations from the absolute value of $ad - bc$, we illustrate it with the data from this example.

$$\chi^2_{(corrected)} = \frac{n(|ad - bc| - .5n)^2}{(a + c)(b + d)(a + b)(c + d)}$$

$$= \frac{100(|24 \times 10 - 36 \times 30| - .5 \times 100)^2}{54 \times 46 \times 60 \times 40}$$

$$= \frac{62,410,000}{5,961,600} = 10.5$$

As you can see, the difference between the two results is not important.

2.8 McNEMAR'S TEST FOR CORRELATED PROPORTIONS

The chi-square test we just considered tests the hypothesis that the proportions estimated from two independent samples are equal. In this section we present a chi-square test for the situation when samples are matched—that is, they are not independent. Investigators frequently use a before-and-after design in which they are trying to test whether there has been a significant change between the before-and-after situations. The features of such a design are illustrated with an example of data on seat belt use before and after a driver was involved in an auto accident. This design, with the data, is shown in Table 12.11.

The appropriate test statistic to use to test the H_0 that there is no change in seat belt use from the period before the accident occurred to that after the accident occurred is **McNemar's chi-square test:**

Table 12.11 A 2 × 2 Table of Seat Belt Use *Before* and *After* Involvement in an Auto Accident for a Sample of 100 Accident Victims

		Wore seat belt regularly after the accident		
		Yes	No	
Wore seat belt regularly before the accident	Yes	$a = 60$	$b = 6$	66
	No	$c = 19$	$d = 15$	34
Total		79	21	100

$$\chi^2 = \frac{(b - c)^2}{b + c}$$

Using the data from Table 12.11, we find that the test shows:

$$\chi^2 = \frac{(6 - 19)^2}{6 + 19} = \frac{169}{25} = 6.76$$

Because the computed $\chi^2 = 6.76$ is larger than the critical value, $\chi^2 = 3.84$, for $\alpha = .05$ with 1 df, we reject the H_0 of no change and conclude that there is a possible increase in seat belt use after involvement in an auto accident. Note that we use only the drivers who have changed in their seat belt use (b and c) in computing McNemar's test.

12.9   MEASURES OF STRENGTH OF ASSOCIATION

A popular measure of the strength of an association between two variables is **relative risk** (RR). Relative risk is widely used in research by clinicians and epidemiologists, largely because it is easy to calculate and interpret.

Relative risk is defined as the ratio of the incidence rate for persons exposed to a risk factor to the incidence rate for those not exposed to the risk factor:

$$\text{Relative risk (RR)} = \frac{\text{incidence rate among exposed}}{\text{incidence rate among unexposed}}$$

Some also call it the risk ratio. We can use a generalized 2 × 2 table to represent frequencies for each of the four cells in a table (Table 12.12). Relative risk can be computed using equation 12.3:

$$\text{RR} = \frac{a/(a + b)}{c/(c + d)} \tag{12.3}$$

Table 12.12 A 2 × 2 Table for Measuring Relative Risk

Risk Factor	Disease Present	Disease Absent	Total
Present	a	b	$a + b$
Absent	c	d	$c + d$

Another commonly used measure of strength of association is the **odds ratio** (OR). The odds ratio, sometimes called relative odds, receives wide use in case-control studies and is defined as the ratio of a/b to c/d. Although OR is not based on disease rates, it is a valid measure of strength of association.

■ **EXAMPLE 4**

In a group of retirees, a community health survey revealed the relationship shown in Table 12.13 between smoking and presence of heart disease. Using the notation of Table 12.12, the relative risk (RR) of developing heart disease is

$$RR = \frac{a/(a + b)}{c/(c + d)} = \frac{25/35}{14/65} = 3.3$$

Based on the results of this survey, smokers have approximately 3.3 times the risk of developing heart disease as nonsmokers. ■

Table 12.13 A 2 × 2 Table of Smoking History and Heart Disease

Risk Factor	Heart Disease		
	Present	Absent	Total
Smoker	25	10	35
Nonsmoker	14	51	65
Total	39	61	100

■ **EXAMPLE 5**

In a controversial study of the relationship between coffee consumption and pancreatic cancer, MacMahon et al. (1981) interviewed 369 cancer patients and 644 controls. Their findings, in part, showed that the patients were much more likely than the controls to have been heavy coffee drinkers. The data are shown in Table 12.14.

The relative odds ratio is computed, using the data from Table 12.14, as

Table 12.14 A 2 × 2 Table for Measuring Relative Odds

Coffee Drinking (cups per day)	Male Pancreatic Cancer Patients	Male Controls
≥5	$a = 60$	$b = 82$
0	$c = 9$	$d = 32$

$$OR = \frac{ad}{bc} = \frac{(60)(32)}{(82)(9)} = 2.6$$

We would estimate from these results that habitual heavy coffee use increased the risk of pancreatic cancer in men by a factor of 2.6 relative to men who did not drink coffee. (This finding has still not been confirmed by other studies.) ∎

We use relative risk when we have two binomial variables obtained from prospective (but not retrospective) studies. Relative risk is a highly useful concept because it provides a quantitative measure relating a stimulus variable (e.g., coffee use) to an outcome variable (e.g., pancreatic cancer).

A relative risk of 2.0 would indicate that heavy coffee use is associated with a twofold (100%) increase in the risk of pancreatic cancer, so coffee may be an important etiologic factor in that type of cancer. It is thus clear why relative risk is so popular. It serves as a quantitative measure of risk, a means of drawing inferences of clinical significance, given the important provision that statistical significance has been established.

12.10   LIMITATIONS IN THE USE OF CHI-SQUARE

We previously mentioned that the techniques suggested in this chapter produce values that follow the continuous chi-square distribution. We use discrete data to approximate a continuous distribution. The closeness of the approximation also depends on the frequency size in the various cells of the contingency table. To ensure that the approximation is adequate, we follow a basic rule: The expected frequencies must not be too small. What is "small"? Its definition can vary by the type of chi-square test being performed. However, a general, well-accepted rule is that no expected frequency should be less than 1 and not more than 20% of the cells should have an expected frequency of less than 5. If a contingency table violates this rule, a good technique is to merge ("collapse") some rows or columns to increase the frequencies of some of the cells. If the expected frequencies are too small, we should use *Fisher's exact test*, described later, in Section 14.7.

The chi-square test is very popular because it is easy to calculate. Also, it can be used with a wide variety of applications in the health and medical sciences. Sometimes, however, its frequency of use leads to misuse. A common misapplication, for example, is to compute a χ^2 statistic for data that do not represent independent observations. This happens when one person is included more than once, when a before-and-after experiment is involved, or when multiple responses are recorded for the same person, as in measuring the frequency of decayed or missing teeth. In the last case, there is obviously a lack of independence, because adjacent teeth in someone's mouth are more likely to be affected than are teeth from different mouths. In such a case, independence would be ensured by counting the number of individuals and classifying them according to the number of decayed or missing teeth rather than by simply counting the number of teeth.

If you suspect that your data are suffering from lack of independence, it would be wise to consult an advanced statistics textbook or obtain help from a statistician. Advanced statistics includes a variety of appropriate methods that can solve almost any problem.

Conclusion

Qualitative data may be analyzed by use of a chi-square test. The object of the test is to determine whether the difference between observed frequencies and those expected from a hypothesis are statistically significant. The test is performed by comparing a computed test statistic, χ^2, with a one-tailed critical value found in a chi-square table. The critical value depends on the selected α and on the number of degrees of freedom, the latter reflecting the number of independent differences as computed from the data. The test statistic is computed as the sum of the ratios of squared differences to expected values. As in other tests of significance, if the computed test statistic exceeds the critical value, the null hypothesis is rejected.

Vocabulary List

a priori	expected frequency	qualitative data
categorical data	frequency data	relative risk
chi-square distribution	McNemar's chi-square	Yates continuity
chi-square test	test	correction
contingency table	observed frequency	
enumeration data	odds ratio	

Exercises

12.1 From the Honolulu Heart Study data in Table 3.1, we can develop a number of chi-square tests of association between two factors. The contingency table for one such test is as follows:

Educational Level	Smoker	Nonsmoker	Total
None	4	16	20
Primary	15	17	32
Intermediate	12	12	24
Senior high	1	8	9
Technical school	0	10	10
Total	32	63	95

a. Using $\alpha = .05$, perform the test and determine whether there is an association between the two variables.

b. Observe that the limitations of the test, as discussed in Section 12.10, were violated, thus invalidating the conclusion of a significant association. To correct the problem of small numbers, combine the senior high and technical school groups to make a 2 × 4 table and repeat the test. Does collapsing the groups change the conclusion?

12.2 As in Exercise 12.1, use Table 3.1 as a source for contingency tables. Test them for associations between the two variables:

a. Activity status (levels 1 and 2) and smoking status (smokers and nonsmokers). Use $\alpha = .01$.

b. Activity status (levels 1 and 2) and systolic blood pressure (classify as less than 140 mmHg for group 1 and greater than or equal to 140 mmHg for group 2). Test at $\alpha = .05$. (*Hint:* Use equation 12.2.)

12.3 A study of diet and age at menarche yielded the following information:

Age of Menarche	Egg Consumption			
	Never	Once per Week	2–4 Times per Week	Daily
Low	5	13	8	4
Medium	4	20	14	0
High	11	18	15	0

a. Test, at $\alpha = .05$, the hypothesis of independence of the two variables. (*Hint:* Use equation 12.1.)

b. Because the expected values indicate a violation of the small numbers limitation of the test, recompute by collapsing the two categories "2–4 times per week" and "daily" into a new category: "2–7 times per week." Does the result change your conclusion?

12.4 Perform chi-square tests for significant difference between the two proportions for the following exercises:

a. 11.5

b. 11.10

c. 11.11

12.5 One of the variables considered in *Heartbeat* (a coronary risk reduction program) was age. An important question emerged: Was the age distribution of the participants different from that of the population in the metropolitan statistical area (MSA) where *Heartbeat* was conducted? Perform a chi-square test to answer the question. Use the MSA population age distribution to compute the expected values.

Age Interval	Heartbeat Participants	MSA Population (1970)
25–34	18	140,195
35–44	33	125,363
45–54	54	120,826
55–64	48	98,884
65 and over	35	125,884
Total	188	611,152

12.6 a. How are degrees of freedom (df) computed for a chi-square table?
b. What is the meaning of degree of freedom in the context of a contingency table?
c. What is a typical H_0 for a contingency table?

12.7 a. What is the basis for computing the expected frequencies in a contingency table?
b. How are the expected frequencies computed?

12.8 What circumstances call for the use of McNemar's test rather than a typical χ^2 test?

12.9 Compute the relative odds ratio (OR) for the data in Table 12.13 and interpret it.

12.10 The following table presents data on 100 pregnant women and their smoking status before and after pregnancy. Determine whether there is a relationship between pregnancy and smoking status.

A 2 × 2 Table of Smoking Status Before and After Pregnancy

Before Pregnancy	Smoking Status After Pregnancy		Total
	Smoker	Nonsmoker	
Nonsmoker	5	55	60
Smoker	20	20	40
Total	25	75	100

12.11 A public health screening survey provided the following data on the relationship between smoking and lung cancer:

	Lung Cancer		
Smoking Status	Present	Absent	Total
Nonsmoker	1	6,700	6,701
Smoker	20	3,279	3,299
Total	21	9,979	10,000

Determine the strength of the association between smoking and lung cancer by computing the relative risk (RR) of a smoker's developing cancer.

12.12 Prepare a contingency table for the data on allergies and geographic region given in Exercise 11.17.
   a. At the $\alpha = .01$ level, determine whether there is an association between the rate of allergy complaints and geographic region.
   b. Compare the conclusions reached in (a) with the one from Exercise 11.17. Why are they the same or different?

12.13 A survey of 100 men and 100 women revealed that 15 of the men and 36 of the women were more than 20% overweight. Prepare a contingency table and test the hypothesis that the two sex groups are homogeneous with respect to being overweight. (Use $\alpha = .05$.)

12.14 Prepare a contingency table for the data on gender and fitness given in Exercise 11.18.
   a. Determine whether there is an association between gender and fitness at $\alpha = .05$.
   b. Compare your conclusion with that reached in Exercise 11.18.

12.15 Prepare a contingency table for the data on seat belt use and gender given in Exercise 11.15.
   a. Determine whether the proportion of seat belt users is the same for both sexes by performing the test of homogeneity.
   b. How do your conclusions differ from those reached in Exercise 11.15?
   c. Compute χ^2, using both the equation requiring expected frequencies (equation 12.1) and the one that does not (equation 12.2). How do the results differ?

12.16 A study was done to examine predictors of readiness to change smoking behavior in a predominantly African American community. Barriers to quitting smoking were examined for associations between races. One of the barriers examined was boredom. Residents in the community were asked whether boredom would be a problem, and therefore a barrier to quitting, should the respondent quit smoking. The respondents were divided into two groups—African Americans and whites/others—and the results are shown in the 2 × 2 table that follows. Based on these data, is there a relationship between race and boredom as a barrier to quitting smoking?

	Ethnicity	
Boredom Would Be a Problem If Stopped Smoking	African American $N = 268$	White/Other $N = 111$
Yes	75	51
No	193	60

NOTE: These data were extrapolated and based on the study by I. Tessaro et al. (1997).

12.17 A study investigated the differences between incarcerated juveniles from alcoholic families and those from nonalcoholic families. Three variables examined were substance abuse, family violence, and child neglect. The following three tables were compiled from the data obtained from incarcerated juveniles. Based on these three 2 × 2 tables, analyze the data.

Substance Abuse	High	Low
Alcoholic Family	28	12
Nonalcoholic Family	13	15

Family Violence	Police Called to Home 1 or More Times	No Police Calls
Alcoholic Family	25	15
Nonalcoholic family	6	22

Neglect	Left Alone for Long Periods	Not Left Alone for Long Periods
Alcoholic Family	5	35
Nonalcoholic Family	8	20

NOTE: These data were extrapolated and based on McGaha and Leoni (1995).

12.18 A study investigated dietary differences between low income African American women and low income white women. One dietary practice examined was the consumption of mutagen-containing meats (defined as a serving of any meat that has been smoked, grilled, or fried). Based on the following table, is there any reason to believe that there are differences between low income African American women and low income white women, with respect to their consumption of mutagen-containing meats?

Race	Mutagen Containing Meats		
	0–1 Servings per Day	2–3 Servings per Day	4 or more Servings per Day
African American	68	36	11
White	73	18	4

NOTE: These data were extrapolated and based on Cox (1994).

12.19 A study was conducted to assess the relationship between syphilis and HIV infection in injection drug users in the Bronx, New York. One part of the study examined the relationship between the incidence and prevalence of syphilis and whether or not the drug user was involved with "paid sex." A 2 × 2 table was developed as shown here.

Paid Sex	Syphilis Cases	
	Positive	Negative
Yes	16	137
No	19	618

NOTE: These data were extrapolated and based on Gourevitch et al. (1996).

a. Based on the data in this table, calculate the odds ratio.
b. After calculating the odds ratio, explain the results.

13 Correlation and Linear Regression

Chapter Outline

Learning Objectives

After studying this chapter, you should be able to

1. Distinguish between the basic purposes of correlation analysis and regression analysis

2. Plot a scatter diagram

3. Compute and explain the meaning of a correlation coefficient in terms of
   a. the kind of data it may be used for
   b. the kind of relationship it can measure
   c. its limitations

4. Compute and interpret a regression equation

5. Perform a test of significance of a correlation coefficient and of a regression coefficient

6. Find the confidence limits for ρ and β

13.1   RELATIONSHIP BETWEEN TWO VARIABLES

Some of our most intriguing scientific questions deal with the relationship between two variables. Is there a relationship between underground nuclear explosions and the increased frequency of earthquakes? Does a relationship exist between use of oral contraceptives and the incidence of thromboembolism? What is the relationship of a mother's weight to her baby's birthweight? These are typical of countless questions we pose in seeking to understand the relationship between two variables.

Whenever an unusual event occurs, people speculate as to its cause. There is an all-too-human tendency to attribute a **cause-and-effect relationship** to variables that *might* be related. Innumerable variables appear to be related to other variables but fail as plausible explanations of causal relationships. For instance, there is a significant association between a child's foot size and handwriting ability, but we would hesitate to claim that a large foot causes better handwriting. A more logical explanation is that foot size and handwriting ability both increase with age; thus the relationship is not causal but direct and age-dependent. As another example, one investigator reported a high degree of association between increased washing machine sales and admissions to mental institutions. It would require a rather convoluted argument to demonstrate a causal relationship between these two variables.

Spurious associations between variables have so perplexed scientists that one of them, Everett Edington of the California Department of Education, composed a clever essay, "Evils of Pickle Eating" (Figure 13.1), in which he satirizes such relationships. To see how easily one might be deceived into believing that a cause-and-effect relationship, however ridiculous, exists, just exchange "milk," "candy," or "bread" for "pickle" in Edington's lampoon.

How, then, can we demonstrate the existence of an actual causal relationship? What statistical methods are available to measure the relationship between two variables?

In previous chapters, we dealt exclusively with observations representing one variable. In this chapter, we consider the relationship of two variables, x and y, obtained for individuals or particular phenomena. Such pairs are referred to as **bivariate data.** We discuss the methods of measuring the relation-

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Evils of Pickle Eating

Pickles are associated with all the major diseases of the body. Eating them breeds war and Communism. They can be related to most airline tragedies. Auto accidents are caused by pickles. There exists a positive relationship between crime waves and consumption of this fruit of the cucurbit family. For example . . .

Nearly all sick people have eaten pickles. The effects are obviously cumulative.

- 99.9% of all people who die from cancer have eaten pickles.
- 100% of all soldiers have eaten pickles.
- 96.8% of all Communist sympathizers have eaten pickles.
- 99.7% of the people involved in air and auto accidents ate pickles within 14 days preceding the accident.
- 93.1% of juvenile delinquents come from homes where pickles are served frequently. Evidence points to the long-term effects of pickle eating.
- Of the people born in 1839 who later dined on pickles, there has been a 100% mortality.

All pickle eaters born between 1849 and 1859 have wrinkled skin, have lost most of their teeth, have brittle bones and failing eyesight—if the ills of pickle eating have not already caused their death.

Even more convincing is the report of a noted team of medical specialists: rats force-fed with 20 pounds of pickles per day for 30 days developed bulging abdomens. Their appetites for WHOLESOME FOOD were destroyed.

In spite of all the evidence, pickle growers and packers continue to spread their evil. More than 120,000 acres of fertile U.S. soil are devoted to growing pickles. Our per capita consumption is nearly four pounds.

Eat orchid petal soup. Practically no one has as many problems from eating orchid petal soup as they do with eating pickles.

EVERETT D. EDINGTON

Figure 13.1   An Example of Spurious Associations Between Variables.   SOURCE: "Evils of Pickle Eating," by Everett D. Edington, originally printed in *Cyanograms*.

ships of bivariate data, determine the strength of the relationships, and make inferences to the population from which the sample was drawn.

3.2   DIFFERENCES BETWEEN CORRELATION AND REGRESSION

The two most common methods used to describe the relationship between two quantitative variables (x and y) are **linear correlation** and **linear regression.**

The former is a statistic that measures the *strength* of a bivariate association; the latter is a **prediction equation** that estimates the value of y for any given x.

When should you use correlation and when regression? Your choice depends on the questions raised and the kind of assumptions you make about the data. For example, you may address questions such as "Is there a relationship between IQ and grade-point average? Is there a relationship between the concentration of fluoride in drinking water and the number of cavities in children's teeth?" Such questions are approached by means of the **correlation coefficient,** which is a measure of the strength of the relationship between the two variables, providing the relationship is linear. As we will see in Section 13.4, it is appropriate to compute a correlation coefficient for such data because both x and y may be considered as random variables (i.e., variables that fluctuate in value according to their distribution).

Certain conventions apply to bivariate data. Almost universally, x refers to the **independent** (or **input**) **variable,** because its outcome is independent of the other variable; and y refers to the **dependent** (or **outcome**) **variable** because its response is dependent on the other variable. Suppose you ask, "What change will occur in one's blood pressure after one reduces salt intake?" Here you would use the regression method, because you are interested in the *degree* of relationship between two variables. Blood pressure would be represented by y, the dependent variable; salt intake by x, the independent variable. You can see from this example that the investigator may arbitrarily select the values of the independent variable and then observe the results of the experiment in terms of the dependent variable y for various levels of x.

To further illustrate the methods of correlation and regression, let us suppose you are interested in studying the relationship of the prepregnancy weights of a group of mothers to their infants' birthweights. "How strong," you might ask, "is the association between the mother's weight and her infant's birthweight?" The method of choice is to calculate a correlation coefficent as a measure of the strength of association between these two variables.

On the other hand, if you were to ask, "What would be an infant's predicted birthweight for a mother possessing a known prepregnancy weight?" you would employ linear regression analysis.

13.3   THE SCATTER DIAGRAM

An ever-popular graphical method used to display the relationship between two variables is the **scatter diagram** (or **scattergram**). The scatter diagram plots the value of each pair of bivariate observations (x, y) at the point of intersection, respectively, of the vertical line through the x value on the abscissa and of the horizontal line through the y value on the ordinate. For instance, let us use data from the Loma Linda Fetal Alcohol Syndrome study (Kuzma and Sokol, 1982), displayed in Table 13.1. We can make a scatter diagram of these data by plotting

Table 13.1 Prepregnancy Weights of Mothers and Birthweights
of Their Infants (Based on Sample Size 25)

Case Number	Mother's Weight (kg)	Infant's Birthweight (g)
1	49.4	3515
2	63.5	3742
3	68.0	3629
4	52.2	2680
5	54.4	3006
6	70.3	4068
7	50.8	3373
8	73.9	4124
9	65.8	3572
10	54.4	3359
11	73.5	3230
12	59.0	3572
13	61.2	3062
14	52.2	3374
15	63.1	2722
16	65.8	3345
17	61.2	3714
18	55.8	2991
19	61.2	4026
20	56.7	2920
21	63.5	4152
22	59.0	2977
23	49.9	2764
24	65.8	2920
25	43.1	2693

SOURCE: Loma Linda Fetal Alcohol Syndrome study.

on a graph each point corresponding to an (x, y) value (Figure 13.2). Take case 13, for example. The mother's prepregnancy weight was 61.2 kg, and she delivered a baby weighing 3062 g. The point appears on Figure 13.2 where the lines for these values intersect. The diagonal line is called the **regression line** or, sometimes, the **line of best fit.** From this line, we expect women weighing 61.2 kg (prepregnancy) to bear babies weighing about 3400 (precisely 3387) g. But we also expect random variation—and, of course, it happens. Case 13's baby weighed 3062 g, 325 g less than would be expected solely on the basis of the mother's weight. This difference is called the **residual.** We will further examine the subject of regression later.

In examining the data of Figure 13.2, you will notice that there is some sort of a relationship between the mother's prepregnancy weight and the infant's birthweight. Although the relationship is subtle, mothers of low prepregnancy weight are seen generally to bear infants of low birthweights, whereas mothers of high prepregnancy weight generally bear heavier infants. Is the relationship linear? An easy way to tell is to examine its scatter diagram to see if the trend
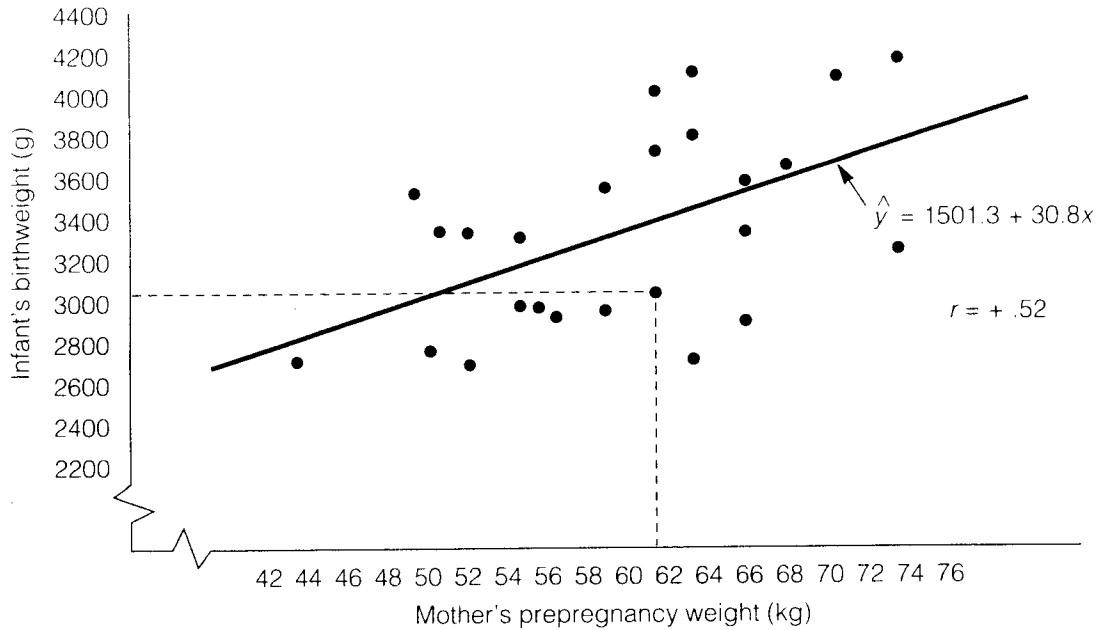
Figure 13.2   Scatter Diagram of Infants' Birthweights Relative to Mothers' Prepregnancy Weights

roughly follows a straight line. How strong is the relationship? To find out, you need to compute an appropriate statistic, such as the correlation coefficient.

13.4   THE CORRELATION COEFFICIENT

As we noted earlier, the sample correlation coefficient, r, is a measure of the strength of the linear association between two variables, x and y. The population value is given by ρ (rho). The correlation coefficient is often referred to as Pearson's product-moment r. It has some unique characteristics: It may take on values between -1 and $+1$, and it is a pure number and nondimensional; that is, it has no units such as centimeters or kilograms. A correlation coefficient of zero represents no relationship between the variables. The closer the coefficient comes to either $+1$ or -1, the stronger is the relationship and the more nearly it approximates a straight line. A **positive correlation** implies a direct relationship between the variables, and a **negative correlation** implies an inverse relationship.

The sample correlation coefficient is defined by

$$r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\sqrt{[\Sigma(x - \bar{x})^2][\Sigma(y - \bar{y})^2]}} \tag{13.1}$$

In computing, we more often use

$$r = \frac{\Sigma xy - \dfrac{(\Sigma x)(\Sigma y)}{n}}{\sqrt{\left[\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}\right]\left[\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}\right]}} \tag{13.2}$$

Another formula, mathematically equivalent but easier to remember because it is defined in terms of the means and standard deviations of x and y and S_{xy}, the **sample covariance** of x only, is

$$r = \frac{\Sigma xy - n\bar{x}\bar{y}/(n-1)}{S_x S_y} = \frac{S_{xy}}{S_x S_y} \tag{13.3}$$

Figure 13.3 illustrates six quite different sets of data and how they are summarized by r. Figure 13.3a illustrates the case of $r = +1.0$, a perfect positive correlation in which all the points fall on a straight line. It is positive because the values of y increase with increases in x. Figure 13.3b is a perfect negative correlation of $r = -1.0$. All the points again fall on a straight line, but as x increases, y decreases.

In real life, there are always random variations in our observations; hence, a perfect linear relationship is extremely rare. Some examples of positive relationships are height and weight, IQ and grade-point average, cigarette consumption and heart disease risk. A negative correlation would describe the relationship between the concentration of fluoride in drinking water and the prevalence of cavities in children's teeth.

Although it is no longer 1.0, the correlation coefficient remains high when the points cluster fairly closely around a straight line (Figure 13.3c). The coefficient becomes smaller and smaller as the distribution of points clusters less closely around the line (Figure 13.3d), and it becomes virtually zero (no correlation between the variables) when the distribution approximates a circle (Figure 13.3e). Figure 13.3f illustrates one drawback of the correlation coefficient: It is ineffective for measuring a relationship that is not linear. In this case, we observe a neat curvilinear relationship whose linear correlation coefficient is quite low. This situation occurs because linear correlation tells its user how closely the relationship follows a straight line.

It is useful to know that the value of r does not change if the units of measurement of a particular variable change. For example, the value of r remains the same whether the measurements are inches and pounds or centimeters and kilograms. Also, r^2 provides an estimate of the proportion of the total variation in the variable y that is explained by the variation in the variable x.

To illustrate the computation of a correlation coefficient, we can apply the
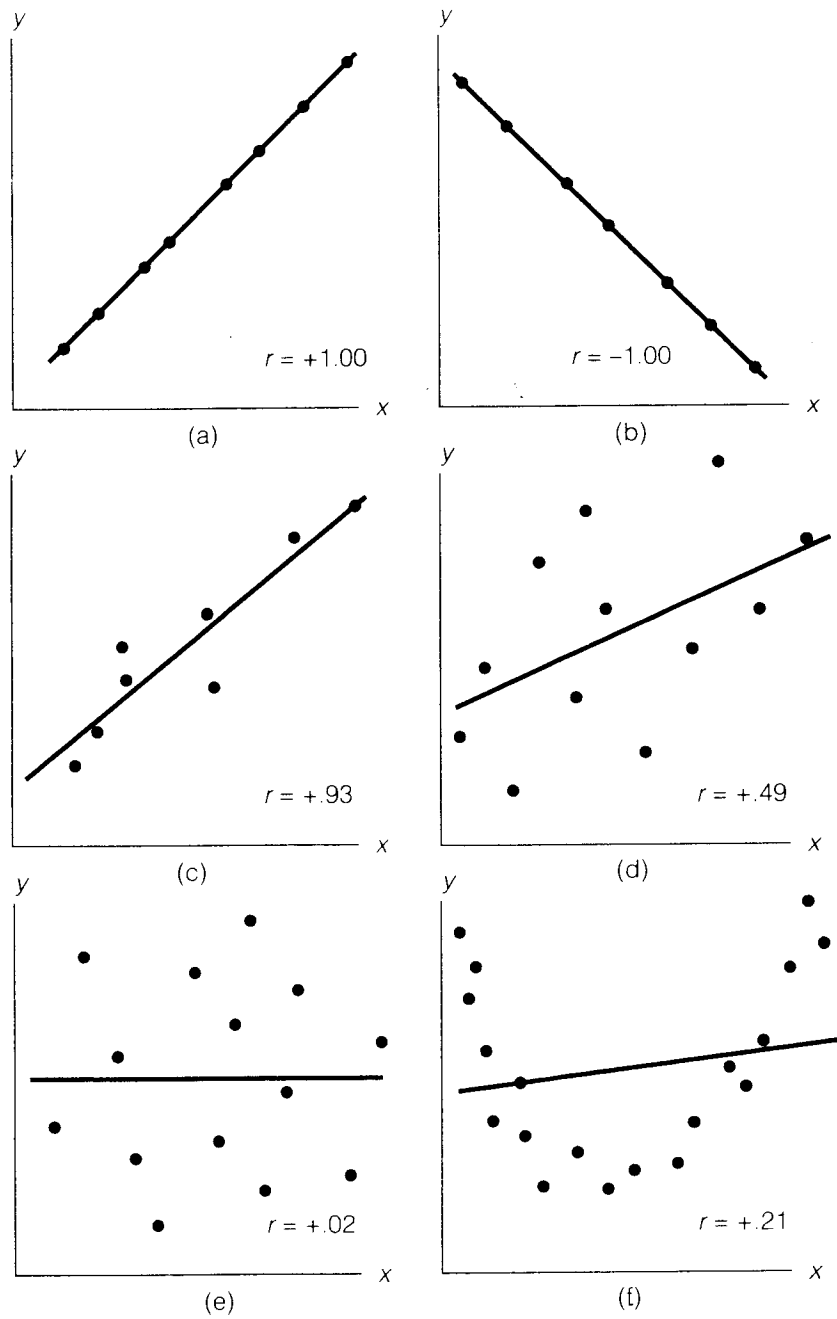
Figure 13.3   Examples of Various Values of *r*

data of Table 13.1. Using equation 13.2, we obtain

$$r = \frac{\Sigma xy - \dfrac{(\Sigma x)(\Sigma y)}{n}}{\sqrt{\left[\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}\right]\left[\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}\right]}}$$

$$= \frac{5{,}036{,}414 - \dfrac{(1494)(83{,}530)}{25}}{\sqrt{\left[90{,}728 - \dfrac{(1494)^2}{25}\right]\left[284{,}266{,}104 - \dfrac{(83{,}530)^2}{25}\right]}} = .51615$$

A correlation coefficient of .51615 seems to be of moderate magnitude. But to interpret it, we need to answer two questions: What inferences can we make regarding its true value? Is the correlation statistically significant?

Curvilinear Relationships

If the scatter diagram indicates that the data do not fit a linear model then the relationship may be **curvilinear,** such as shown in Figure 13.3(f). It would not make much sense to try to fit a least squares line in such a situation. One possible solution would be fitting a linear regression to a transformed set of variables such as $\sqrt{y}$. If the error terms are smaller using $\sqrt{y}$ then we have gained some in keeping a simple straight-line model to explain the relationship. There are a number of different transformations that could be used such as y^2, $\frac{1}{y}$, or $\log y$. The object is to obtain a better linear relationship than the original data. However, there are no precise ways to determine which transformation one should use.

Coefficient of Determination

A definition for $r^2 = 1 - \left(\frac{SSE}{SST}\right)$, where SST represents the total sum of squares and SSE is the sum of squares $\Sigma(y - \bar{y})^2$, which represents the overall variability of the response variable y.

We should note the following characteristics about r^2:

a. It is always between 0 and 1. At the extreme value of 0, the regression line is horizontal; that is, $b_1 = 0$.

b. The closer r^2 is to 1, the "better" the regression line is in the sense that the residual sum of squares is much smaller than the total sum of squares. For this reason the r^2 is usually reported as an overall "figure of merit" for regression analysis.

We can interpret r^2 as the fraction of the total variation in y (SST) that is accounted for by the regression relationship between y and x.

13.5   TESTS OF HYPOTHESES AND CONFIDENCE INTERVALS FOR A POPULATION CORRELATION COEFFICIENT

As you might expect, the correlation coefficient r is a simple value. It is an estimate of the population correlation coefficient ρ in the same sense that x is an estimate of the population mean μ. We are most often interested in drawing inferences from a sample to the general population, so it is logical to perform a test of significance on the population correlation coefficient and estimate a confidence interval for it.

If you wish to test the null hypothesis that $\rho = 0$ (i.e., x and y are not linearly correlated) against the alternative hypothesis that $\rho \neq 0$, you can use the following procedure. The only needed assumptions: the pairs of observations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ must have been obtained randomly, and both x and y must be normally distributed. The test statistic to use is

$$t = \frac{r - 0}{\sqrt{(1 - r^2)/(n - 2)}} \tag{13.4}$$

with $n - 2$ df, where n is the number of paired observations.

For our mother–child example,

$$t = \frac{.51615}{\sqrt{[1 - (.51615)^2] \cdot (25 - 2)}} = 2.89$$

which (by reference to the t table) represents a correlation significantly ($\rho < .01$) different from zero. Our conclusion: There appears to be a positive association between a woman's prepregnancy weight and her infant's birthweight. Very often, in a journal article, the researchers will have performed multiple correlations. The correlations will then be displayed in a table that is often referred to as a correlation matrix. Table 13.2 is an abbreviated version of an actual correla-

Table 13.2

Variable	1	2	3	4	5	6	7	8
1	—							
2	.07	—						
3	.27**	.16**	—					
4	.23**	.34**	.35**	—				
5	.03	−.11**	.33**	.16**	—			
6	−.05	−.07	.15**	.11**	.32**	—		
7	.05	.36**	−.11**	.16**	−.22**	−.03	—	
8	.08*	−.07	.30**	.22**	.30**	.12**	−.19**	—
$N = 733$		*$p < .05$	**$p < .01$					

NOTE: These data were copied and the table abbreviated from Windle and Windle (1996).

tion matrix created by Windle and Windle (1996). The complete matrix included all possible correlations from 15 variables. The original matrix had a total of 105 correlation coefficients!

For each of the correlation coefficients, we have used a computer to compute a t test for significance. This matrix is a typical display of the correlation coefficients and asterisks to indicate which correlations are significant at .05 and .01.

Notice in Table 13.2 that the correlation of variable 1 and variable 8 yields a correlation coefficient of .08. This correlation is significant at .05. Under just about any conceivable circumstance, .08 is a very low correlation, yet in this study was found to be significant. How is it possible that such a low correlation could be significant? The answer has to do with the sample size. When a sample is large (733 is a large sample), using equation 13.4 will almost always give you a significant correlation. Note what happens to the calculated t with a correlation coefficient of .08 and an N of 733.

$$t = \frac{.08}{\sqrt{1 - (.08)^2/731}}$$

The calculated t of 2.17, when compared to a critical t of ± 1.97 (200 df from Table B) yields a statistically significant correlation. The inescapable conclusion is that you must be careful about interpreting the meaning of a significant correlation when the sample size is large.

Where did the t statistic of equation 13.4 come from? Mathematical statisticians are able to make a comparatively simple derivation from other equations, as you will see in Section 13.8.

Computing a confidence interval for ρ involves an equation much more complex than the corresponding one for the population mean. In consequence, tables giving **confidence intervals** have been prepared for the convenience of the user. Figure 13.4 illustrates 95% confidence intervals for different sample sizes. Suppose you wanted to find the 95% confidence interval for the population correlation coefficient ρ from the mother–child example ($r = .52, n = 25$). It is quite simple to do this by using Figure 13.4. Find the r of $+.52$ on the abscissa and sketch a vertical line through it. The points given by the intersection of that line and the intervals for $n = 25$ give the upper and lower 95% confidence limits. Use the curves that correspond to your sample size or visually interpolate. The limits are read on the ordinate, approximately $+.10$ and $+.75$. If we can safely assume that our data for the 25 mother–child pairs (Table 13.1) are a random sample of all the pairs in the study, then the 95% confidence interval for the true population ρ is indeed .10 to .75. Regardless of the true value of the population correlation coefficent, we can draw the inference, with 95% confidence, that it is captured by the range .10–.75. Further, the test of the H_0: $\rho = 0$ at the $\alpha = .01$ level indicates that ρ is significantly different from zero because zero falls below the interval .10–.75.

Scale of r (= sample correlation coefficient)
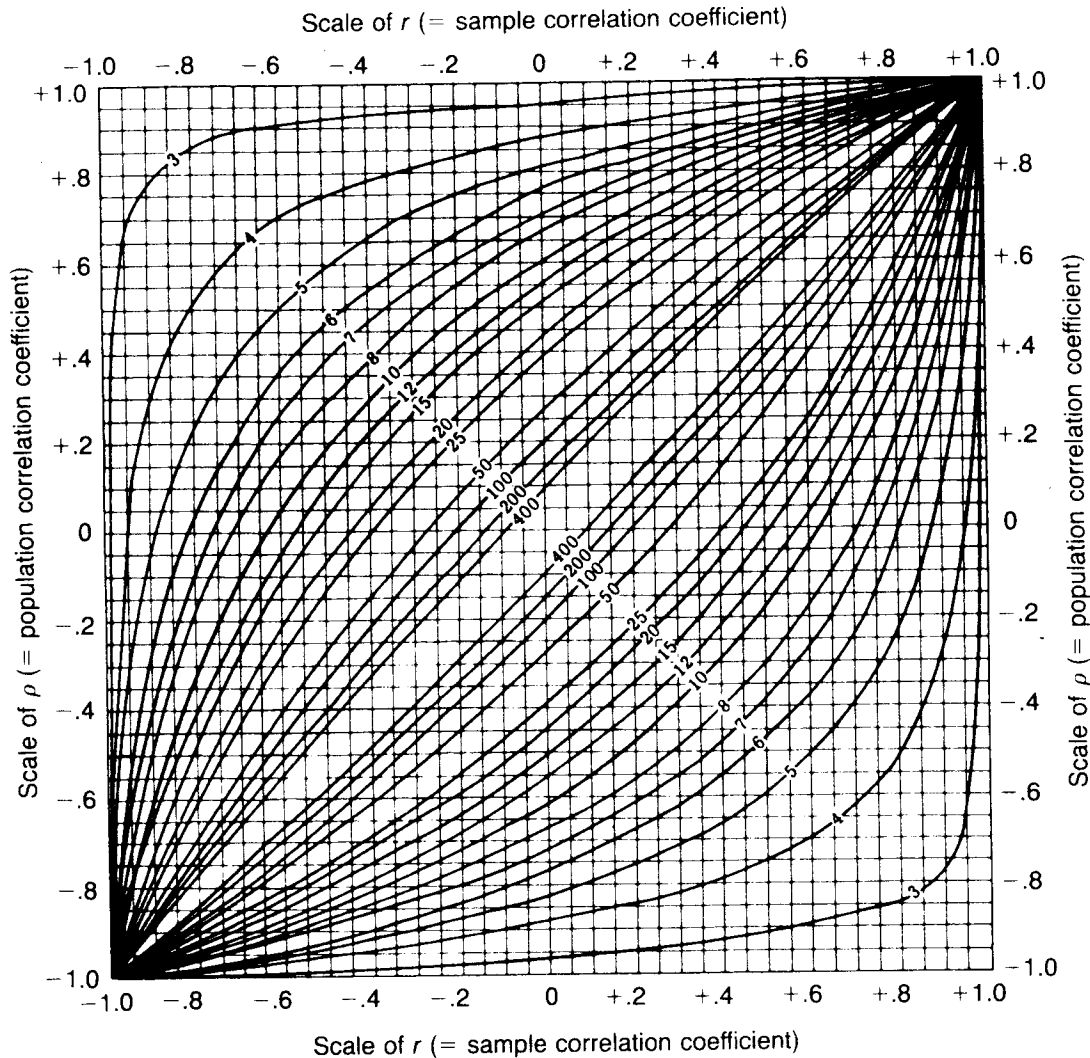
Figure 13.4 Confidence Intervals for the Correlation Coefficient $(1 - \alpha = .95)$. SOURCE:
Reprinted with permission from *Handbook of Tables for Probability and Statistics*, ed. William
H. Beyer (Boca Raton, Fl.: CRC Press, 1966). Copyright CRC Press, Inc., Boca Raton, Fl.
NOTE: The numbers on the curves are sample sizes.

13.6   LIMITATIONS OF THE CORRELATION COEFFICIENT

As we mentioned, one limitation of the correlation coefficient is that, though it
measures how closely the two variables approximate a straight line, it does not
validly measure the strength of a nonlinear relationship. We also have to equiv-
ocate a bit as to the reliability of the correlation when n is small (say, fewer than
about 50 pairs of observations). Further, it is always useful to plot a scattergram
(e.g., Figure 13.2) to see if there are any **outliers**—that is, observations that
clearly appear to be out of range of the other observations. Outliers have a

marked effect on the correlation coefficient, often suggest erroneous data, and are likely to give misleading results. Perhaps the most important drawback of the correlation coefficient is that a high (or statistically significant) correlation can so easily be taken to imply a cause-and-effect relationship. Use caution: Do not take it as proof of such a relationship.

With all these reservations, you may be puzzled as to how major decisions in public policy can be based on correlation analysis. For instance, in the Surgeon General's Report (U.S. Department of Health, Education, and Welfare, 1971), we see an important public document that includes a good deal of correlation analysis and concludes that smoking causes lung cancer. In reaching their conclusions, the Surgeon General's blue-ribbon panel of experts (which included leading statisticians) relied heavily on the consistency of the results of a large number of population and laboratory studies. In essence, their conclusion was based not on a single correlation coefficient, but on an overwhelming body of evidence:

1. The death rate for cigarette smokers was about 70% higher than for non-smokers.

2. Death rates increased with increased smoking (Table 13.3).

3. The death rates of heavy smokers were more than two times as large as those of light smokers.

4. The mortality ratio of cigarette smokers to nonsmokers was substantially higher for those who started to smoke under age 20 than for those who started smoking after age 25. The mortality ratio increased with more years of smoking.

5. The mortality of smokers who inhaled was higher than that of those who did not.

6. Persons who stopped smoking had a mortality ratio 1.4 times that of persons who never smoked, while current smokers had a ratio of 1.7.

7. In prospective studies, it was found that for all causes of death, smokers experienced 70% greater mortality than nonsmokers, but for respiratory

Table 13.3 Correlation Between Increased Smoking and Increased Death Rate

No. of Cigarettes Smoked	Mortality Ratio of Smokers to Nonsmokers	Excess in Death Rate of Smokers Over Nonsmokers
<10	1.45	45%
10–19	1.75	75%
20–39	1.90	90%
40 or more	2.20	120%

system causes the percent was even higher. For lung cancer, it was 10 times higher; for bronchitis and emphysema, it was 6.1 times higher.

13.7   REGRESSION ANALYSIS

We are indebted to Sir Francis Galton for coining the term *regression* during his study of heredity laws. He observed that physical characteristics of children were correlated with those of their fathers. He noted particularly that the heights of sons were less extreme than those of their fathers. Specifically, he found that tall fathers tended to have shorter sons, whereas short fathers tended to have taller sons, a phenomenon he called "regression toward the mean." In plotting median heights of sons and fathers, he found that there was a positive association and that the relationship was roughly linear.

Subsequently, statisticians used means, not medians, and embraced the term *regression line* to describe a linear relationship between two variables. The regression line also indicates prediction of the value of a dependent (outcome) variable (y) from a known value of an independent variable (x), and the expected change in a dependent variable for a unit change in an independent variable. For any two variables, there is a linear equation that best represents the relationship between them. It is often useful to find an estimate of the true equation that describes the straight-line regression. Such an estimate is given by

$$\hat{y} = a + bx \tag{13.5}$$

That is, the dependent variable $\hat{y}$ can be estimated in terms of a constant, a, plus another constant, b, times the independent variable x. Note the important distinction between $\hat{y}$, the predicted value (which falls on the regression line), and y, the observed value that usually does not fall on the line. The constants a and b are estimates of the two parameters of the true regression equation that define the location of the line. Their specific meaning is illustrated in Figure 13.5. The constant a represents the value of y when $x = 0$, while b is the *slope* (or *gradient*) of the line. The slope can be more precisely defined as the amount of change, Δy, in the dependent variable for a given change, Δx, in the independent variable. Thus, the slope, often referred to as the **regression coefficient,** gives a good indication of the relationship between variables x and y.

Equation 13.5 is an estimate of the following equation, which describes the population regression of y on x:

$$y = \beta_0 + \beta_1 x + \epsilon$$

where β_0 is the **y-axis intercept** and corresponds to a of equation 13.5; β_1 is the slope of the population regression line and corresponds to b of equation 13.5; and ϵ is the error in the observed value of y for a specified value of x. The error,
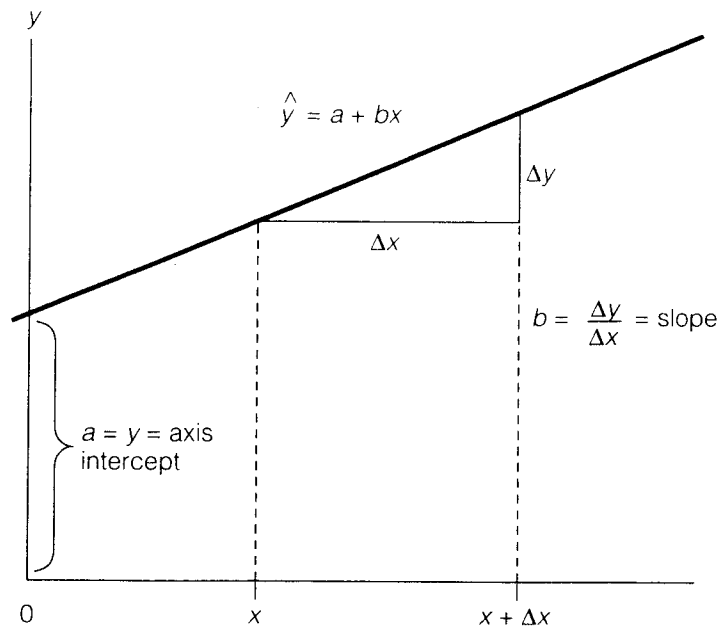
Figure 13.5 Equation of a Straight Line

the residual, is estimated by $y - \hat{y}$, the difference between the observed and the predicted value.

Certainly, you would strive to solve regression problems with some equation that provides the "best fit" to the data. But how would you do this? There is a mathematical procedure that minimizes the estimated error $(y - \hat{y})$. It is known as the **least-squares method.** This procedure uses equations that estimate β_0 and β_1 by the following equations for a and b. The equation for estimating β_1 is

$$\hat{\beta}_1 = b = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{\Sigma(x - \bar{x})^2} = r_{xy}\frac{s_y}{s_x} \qquad (13.6)$$

and the equation for estimating β_0 is

$$\hat{\beta}_0 = a = \bar{y} - b\bar{x} \qquad (13.7)$$

Again using our data on mothers' and infants' weights (Table 13.1), we can now compute the slope. For convenience, we use the mathematically identical computation equations:

$$b = \frac{\Sigma xy - [(\Sigma x)(\Sigma y)]/n}{\Sigma x^2 - [(\Sigma x)^2]/n}$$

$$= \frac{5,036,414.1 - (1,493.7)(83,530)/25}{90,728.45 - (1,493.7)^2/25} = 30.794$$

(13.8)

and

$$a = y - bx = 3341.2 - 30.794 (59.748) = 1501.32 \qquad (13.9)$$

Now that we know the values of the two constants, we can write the equation for the best-fitting line of regression:

$$\hat{y} = 1501.32 + 30.794x$$

Symbolically, $\hat{y}$ is the predicted value for a given value of x. It is actually the estimated mean of all y's that could be observed for a specific value of x.

To illustrate further: Women with a prepregnancy weight of 70.3 kg would be expected from the preceding equation to bear infants weighing an average of 3666 g. But case 6, a subject who weighed 70.3 kg, bore a baby weighing 4068 g. The difference, $y - \hat{y} = 4068 - 3666 = 402$ g, represents the deviation, or residual, of the observed value from the value predicted by the least-squares regression line. The residuals are shown as the vertical lines in Figure 13.6.

The regression line always passes through the means of x and y—that is, through $(x = x, y = y)$. Hence, it is simple to superimpose it on the scattergram. A characteristic of a least-squares regression line is that the sum of the devia-
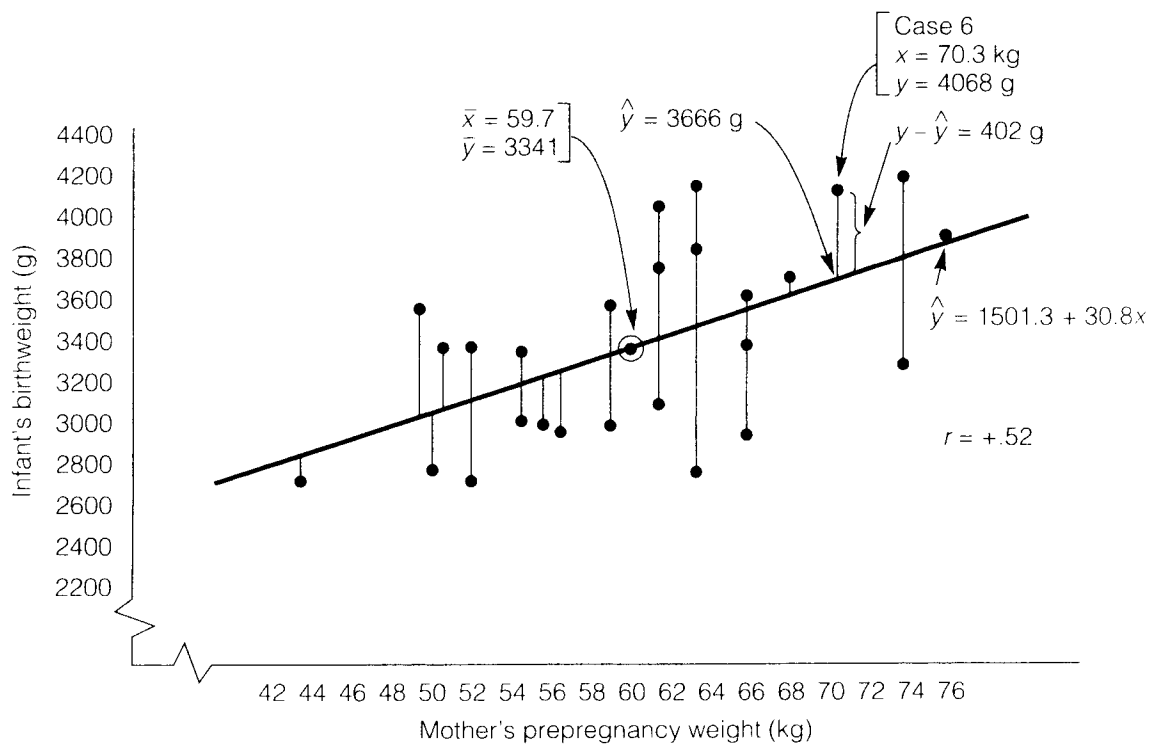
Figure 13.6 Deviations About the Linear Regression Line for Infants' Birthweights Relative to Mothers' Prepregnancy Weights

tions about the line is equal to zero, and the sum of the squared deviations is a minimum; that is, there is no other line for which it could be less. That is why it is referred to as the line of best fit in the sense of "least squares." Table 13.4 helps verify this. It shows that the sum of the residuals above the regression line equals the sum of those below the line; that is, $\Sigma(y - \hat{y}) = 0$ or actually 0.14, which is a tiny roundoff error.

An indication of just how precisely the regression line describes the relationship between x and y is the variance of the deviations $(y - \hat{y})$ about the line. This variance is denoted $s_{y \cdot x}^2$. It is an estimate of the true error of prediction $\sigma_{y \cdot x}^2$. Underlying this estimate is an assumption of homogeneity—namely, that $\sigma_{y \cdot x}^2$ remains constant for all y's distributed about each x along the regression line.

The last column of Table 13.4 is used to compute $s_{y \cdot x}^2$, the equation being

$$s_{y \cdot x}^2 = \frac{\Sigma(y - \hat{y})^2}{n - 2} \tag{13.10}$$

where $n - 2$ represents the degrees of freedom. From Table 13.4, we compute $s_{y \cdot x}^2$ as $3{,}769{,}490.7/23 = 163{,}890.9$. Alternatively, the same variance can be obtained directly without computing predicted values $(\hat{y})$ by substituting $\alpha + bx$ for y, which gives

$$s_{y \cdot x}^2 = \frac{\Sigma(y - a - bx)^2}{n - 2} \tag{13.11}$$

After some algebraic manipulations this can be rewritten as

$$s_{y \cdot x}^2 = \frac{\Sigma y^2 - a\Sigma y - b\Sigma xy}{n - 2} \tag{13.12}$$

The square root of $s_{y \cdot x}^2$ is referred to as the **standard error of estimate.** Once you have obtained the equation for a linear regression line, you would probably like to know how reliable the line is for predicting dependent variables. To find out, you need to use the standard error of estimate in a test of significance or obtain confidence intervals for β_1, the slope of the population regression line.

.8 INFERENCES REGARDING THE SLOPE OF THE REGRESSION LINE

Thus far we have assumed that (1) the means of each distribution of y's for a given x fall on a straight line, and (2) the variances, $\sigma_{y \cdot x}^2$, are homogeneous for each distribution of y's for a given x. To perform tests of significance or compute confidence intervals, we will need one more assumption: The distribution of y's is normal for each value of x.

Table 13.4 Prepregnancy Weights of Mothers and Birthweights of Their Infants—Deviations About the Linear Line of Regression

Case Number	x Mother's Weight (kg)	y Infant's Actual Birthweight (g)	$\hat{y}$ Infant's Expected Birthweight (g)*	$y - \hat{y}$ Residual (g)	$(y - \hat{y})^2$ Squared Residual (g)²
1	49.4	3515	3022.54	492.46	242,516.85
2	63.5	3742	3456.73	285.27	81,378.97
3	68.0	3629	3595.31	33.69	1,135.02
4	52.2	2680	3108.76	-428.76	183,835.13
5	54.4	3006	3176.51	-170.51	29,073.66
6	70.3	4068	3666.13	401.87	161,499.49
7	50.8	3373	3065.65	307.35	94,464.02
8	73.9	4124	3776.99	347.01	120,415.94
9	65.8	3572	3527.56	44.44	1,974.91
10	54.4	3359	3176.51	182.49	33,302.60
11	73.5	3230	3764.67	-534.67	285,872.00
12	59.0	3572	3318.16	253.84	64,434.75
13	61.2	3062	3385.91	-323.91	104,917.68
14	52.2	3374	3108.76	265.24	70,352.26
15	63.1	2722	3444.42	-722.42	521,890.65
16	65.8	3345	3527.56	-182.56	33,328.15
17	61.2	3714	3385.91	328.09	107,643.04
18	55.8	2991	3219.62	-228.62	52,267.10
19	61.2	4026	3385.91	640.09	409,715.21
20	56.7	2920	3247.33	-327.33	107,144.92
21	63.5	4152	3456.73	695.27	483,400.37
22	59.0	2977	3318.16	-341.16	116,390.14
23	49.9	2764	3037.93	-273.93	75,037.64
24	65.8	2920	3527.56	-607.56	369,129.15
25	43.1	2693	2828.54	-135.54	18,371.09
	$\bar{x} = 59.7480$	$\bar{y} = 3341.20$	—	$\Sigma(y - \hat{y}) = 0.14$	$\Sigma(y - \hat{y})^2 = 3,770,704.90$

*$\hat{y} = 30.794x + 1501.312$.

We noted earlier that the slope, b, computed from sample data is an estimate of some true value, β_1, for the population regression line, which is defined by

$$y = \beta_0 + \beta_1 x + \epsilon \tag{13.13}$$

We now wish to determine (1) how useful the regression line obtained from sample data is in predicting the outcome variable, and (2) whether the slope b differs significantly from $\beta_1 = 0$. To do this, we need to perform a hypothesis test for β much as we did for μ. The first step is to compute the standard error of b. Mathematical statisticians have shown that

$$SE(b) = \sqrt{\frac{s_{y \cdot x}^2}{\Sigma (x - \bar{x})^2}} \tag{13.14}$$

which simplifies to

$$SE(b) = \frac{s_{y \cdot x}}{s_x \sqrt{n - 1}}$$

which for our data on mothers' weights and infants' birthweights (Table 13.4) computes to

$$SE(b) = \frac{\sqrt{163,890.9}}{\sqrt{90,728.45 - 1493.7^2/25}} = \frac{404.834}{38.51} = 10.512$$

Using this value, we can now perform the following hypothesis test:

1. H_0: $\beta_1 = 0$ (slope of 0 means that there appears to be no relationship be-
   tween x and y) versus H_1: $\beta_1 \neq 0$.
2. $\alpha = .05$.
3. The test statistic (with $n - 2$ df) is

$$t = \frac{b - 0}{SE(b)}$$

$$= \frac{30.794}{10.512} \tag{13.15}$$

$$= 2.93$$

4. The critical region for t with 23 df for $\alpha = .05$ is $t = 2.07$.
5. We reject H_0 because a t of 2.93 falls in the critical region.

6. We conclude that the slope differs significantly from 0; consequently, a regression line estimated from our data can, with reasonable reliability, predict dependent variables for given values of x.

From the test statistic for the regression coefficient b, it is a simple matter to describe the confidence interval for the true regression coefficient β_1:

$$\text{CI for } \beta_1 = b \pm t[\text{SE}(b)] \tag{13.16}$$

again based on $n - 2$ df.

The confidence interval corresponds to the central $(1 - \alpha)$ proportion of the area. Assuming only that our data for 25 mother–infant pairs is a random sample of all such pairs, the 95% confidence interval for β_1, the true slope, is

$$\text{CI} = 30.794 \pm 2.07(10.512)$$
$$= 30.794 \pm 21.760$$
$$= 9.03 \text{ to } 52.55$$

Therefore, we can say (with 95% confidence) that β_1 is unlikely to be less than 9.03 or larger than 52.56.

We can show that the t statistics of equation 13.4 can be derived from testing the null hypothesis that β, the slope of the line of regression, is zero. The equation to use is

$$t = \frac{b}{\text{SE}(b)}$$

where b represents a sample estimate of β.

Testing whether β equals zero is functionally equivalent to testing for ρ equals zero. For further details, see Armitage (1971).

Conclusion

Correlation analysis and regression analysis have different purposes. The former is used to determine whether a relationship exists between two variables and how strong that relationship is. The latter is used to determine the equation that describes the relationship and to predict the value of y for a given x. An aid to visualizing these concepts is the scatter diagram.

A correlation coefficient (r) can take on values from -1 to $+1$. The closer r approaches -1 or $+1$, the stronger the linear relationship between x and y; the closer r approaches zero, the weaker the relationship. It is important to keep in mind that a high correlation merely indicates a strong association between the variables; it does not imply a cause-and-effect relationship. A correlation coeffi-

cient is valid only where a linear relationship exists between the variables. After computing the correlation coefficient r and the regression coefficient b, we are obliged to test their significance or set up confidence limits that encompass the population values they estimate.

Vocabulary List

bivariate data
cause-and-effect
    relationship
coefficient of
    determination
confidence intervals
correlation coefficient
curvilinear regression
dependent variable
    (outcome variable)

independent variable
    (input variable)
least-squares method
linear correlation
linear regression
negative correlation
outlier
positive correlation
prediction equation
regression coefficient
    (slope, gradient)

regression line (line of
    best fit)
residual
sample covariance
scatter diagram
    (or scattergram)
standard error of
    estimate
y-axis intercept

Exercises

13.1 A correlation coefficient r consists of two parts: a sign and a numerical value.
  a. What is the range of values possible for r?
  b. What does the sign tell you about the relationship between variables x and y?
  c. What information do you derive from the value of r regarding x and y?
  d. What does r tell you about the ability of the regression line to predict values of y for given values of x?
  e. For any given set of data, would the correlation coefficient and the regression coefficient necessarily have the same sign? The same magnitude?

13.2 For the data of Table 3.1 (Honolulu Heart Study), compute the correlation coefficient for
  a. Blood glucose (x) and serum cholesterol (y), with $\Sigma x = 15,214$; $\Sigma y = 21,696$; $\Sigma x^2 = 2,611,160$; $\Sigma xy = 3,371,580$; $\Sigma y^2 = 4,856,320$.
  b. Ponderal index (x) and systolic blood pressure (y), with $\Sigma x = 13,010$; $\Sigma y = 4,052$; $\Sigma x^2 = 1,736,990$; $\Sigma xy = 527,185$; $\Sigma y^2 = 164,521$. What does this correlation coefficient tell you about the scatter diagram of systolic blood pressure versus ponderal index?

13.3 In a study of systolic blood pressure (SBP) in relation to whole blood cadmium (Cd) and zinc (Zn) levels the following data were obtained:

Cd (ppm/g ash)	68	63	56	48	96	70	66	45	50	60	53	47	36	65
Zn (ppm/g ash)	127	118	78	76	181	134	122	87	80	107	116	103	64	123
SBP (mmHg)	166	162	116	120	160	120	182	134	130	116	108	134	116	96

a. Make a scatter diagram of cadmium and systolic blood pressure, using the latter as the dependent variable.
b. Judging from the diagram, would you be justified in using linear regression analysis to determine a line of best fit for cadmium and blood pressure? Why or why not?
c. Compute the correlation coefficient for cadmium and blood pressure.
d. Using zinc as the dependent variable, plot a scatter diagram of cadmium and zinc.
e. Does the diagram of (d) provide justification for using regression analysis to determine a line of best fit? Why or why not?
f. Calculate the equation of the line of best fit for the relationship between zinc and cadmium, and draw the line on the scatter diagram for (d).
g. If it were determined that a patient had a whole blood cadmium level of 80, what would you expect that patient's zinc level to be?
h. Would you be justified in stating that there is a cause-and-effect relationship between cadmium and zinc? Why or why not?

13.4   Test the correlation coefficient you calculated in Exercise 13.2a to determine if it is significantly different from zero.

13.5   a. Determine the 95% confidence limits for the population correlation coefficient ρ of cadmium and blood pressure for Exercise 13.3c.
b. Test the hypothesis H_0: $\rho = 0$ by using the confidence interval you found in (a).

13.6   To find the equation of the regression line in Exercise 13.3f, you had to calculate the regression coefficient β_1. Perform a significance test of the null hypothesis that the population regression coefficient is not significantly different from zero.

13.7   Calculate the equation of the regression line for the relationship between blood glucose (x) and serum cholesterol (y) for the summary data given in Exercise 13.2a. Perform a test of significance of the H_0: $\beta_1 = 0$ at the $\alpha = .01$ level.

13.8   Calculate the equation of the regression line for the relationship between ponderal index (x) and systolic blood pressure (y) for the summary data given in Exercise 13.2b. Perform a test of significance of the H_0: $\beta_1 = 0$ at the $\alpha = .05$ level.

13.9   What are the assumptions that one needs to make in
a. testing H_0: $\rho = 0$
b. testing H_0: $\beta = 0$ or computing the CI for β

13.10   Give examples of variables that would be suitable for computing a
a. correlation coefficient
b. regression line

13.11   What are the limitations of a correlation coefficient?

13.12   What is the meaning of
a. r
b. r^2
c. a and b in a regression line
d. least-squares regression line

13.13 The following are data for 12 individuals' daily sodium intake and their systolic blood pressure readings.

Person	Sodium	BP	Person	Sodium	BP
1	6.8	154	7	7.0	166
2	7.0	167	8	7.5	195
3	6.9	162	9	7.3	189
4	7.2	175	10	7.1	186
5	7.3	190	11	6.5	148
6	7.0	158	12	6.4	140

A research investigator is interested in learning how strong the association is between these variables and how well we can predict blood pressure from sodium intake.
a. Compute r and test the H_0: $\rho = 0$ at the $\alpha = .05$ level.
b. Obtain the 95% CI for ρ.

13.14 a. Calculate the regression equation for the data in Exercise 13.13.
b. Test the H_0: $\beta = 0$ at the $\alpha = .01$ level.
c. What would be a likely blood pressure for a person with a sodium intake of 6.3? of 7.6?

13.15 Richard Doll, a British investigator of the relationship between smoking and lung cancer, compiled the following information on per capita cigarette consumption in 1930 and lung cancer 20 years later (in 1950) for a number of countries, as shown below:

Country	Cigarette Consumption in 1930	Deaths per 100,000 in 1950
USA	1300	20
Great Britain	1100	46
Finland	1100	35
Switzerland	510	25
Canada	500	15
Holland	490	24
Australia	480	18
Denmark	380	17
Sweden	300	11
Norway	250	9
Iceland	230	6

a. Construct a scatter diagram and describe the relationship between cigarette consumption in 1930 and lung cancer in 1950.
b. Compute r and r^2 and describe what they mean.
c. Test the H_0 that there is no association between cigarette consumption and the subsequent development of lung cancer.

13.16 The American Heart Association has provided the following regression equations for computing a person's ideal weight ($\hat{y}$) based on a person's height (x) in feet. For females it is given by $\hat{y} = 100 + 4.0x$, and for males it is given by $\hat{y} = 110 + 5.0x$. Use the appropriate equation to determine your ideal weight and compare it with your actual weight to determine whether or not you are over- or underweight.

13.17 You obtained a Pearson r of -1.04. What does this tell you about the relationship between the two variables correlated?

13.18 You obtained a Pearson r of .45. How many pairs of subjects or scores must you have for this correlation to be considered significant? Assume that this is a one-tailed test.

13.19 The correlation matrix from which Table 13.2 is derived actually had 105 correlations from the 15 variables. If there were absolutely no significant correlations between any of the variables, at a .05 level of significance, how many correlations would you expect to be significant? (*Hint:* This is directly related to the possibility of a Type I or Type II error.)

13.20 Which of these correlations is the strongest? the weakest? Explain.
   a. $-.71$ b. .08   c. .62   d. $-.12$

13.21 You obtained the following correlation matrix:

A.	1.00	.48	$-.06$	.87	
B.		1.00	.17	$-.71$	$N = 32$
C.			1.00	$-.40$	
D.				1.00	

   a. Which correlations are significant at .05? (two-tail)
   b. Which correlations are significant at .01? (two-tail)

13.22 You obtained a Pearson r of .60.
   a. With an n of 25, 50, and 100, what are the confidence intervals?
   b. Why do the confidence intervals become narrower as the sample size increases?

14 Nonparametric Methods

Chapter Outline

Learning Objectives

After studying this chapter, you should be able to

1. Distinguish between
   a. parametric and nonparametric methods
   b. rank-sum tests and signed-rank tests
   c. Pearson and Spearman correlation coefficients

2. List the advantages and disadvantages of nonparametric methods

3. Give the equation for the sum of the first n integers

4. List the assumptions necessary to perform hypotheses tests by nonparametric methods

5. Be able to apply the sign test to paired data

6. Know when and how to use Fisher's exact test

14.1   RATIONALE FOR NONPARAMETRIC METHODS

In the preceding chapters we discussed several methods that enable us to determine whether there is a significant difference between two sample means. The most popular of these involve the normal and the t distributions. We also learned about the correlation coefficient, which measures the amount of linear association between two variables. Underlying such test statistics were assumptions of normality, homogeneity of variances, and linearity. Whenever we dealt with measurement data used in test statistics, we also were interested in obtaining some estimate of the population parameter—that is, μ or ρ.

All these statistical techniques are collectively referred to as **parametric methods.** In contrast to these are the **nonparametric methods,** which have been developed for conditions in which the assumptions necessary for using parametric methods cannot be made. Nonparametric methods are sometimes referred to as **distribution-free methods** because it is not necessary to assume that the observations are normally distributed. A nonparametric method is appropriate for dealing with data that are measured on a nominal or ordinal scale (discussed in Chapter 1) and whose distribution is unknown. Because of the many advantages of nonparametric methods, their use has been increasing rapidly. But, like most methods, they also have disadvantages.

14.2   ADVANTAGES AND DISADVANTAGES

Nonparametric methods have three main advantages:

1. They do not have such restrictive assumptions as normality of the observations. In practice, data are often nonnormal or the sample size is not large enough to gain the benefit of the central limit theorem. At most, the distribution should be somewhat symmetrical. This gives nonparametric methods a major advantage.

2. Computations can be performed speedily and easily—a prime advantage when a quick preliminary indication of results is needed.

3. They are well suited to experiments or surveys that yield outcomes that are difficult to quantify. In such cases, the parametric methods, although

statistically more powerful, may yield less reliable results than the non-parametric, which tend to be less sensitive to the errors inherent in ordinal measurements.

There are also three distinct disadvantages of nonparametric methods:

1. They are less efficient (i.e., they require a larger sample size to reject a false hypothesis) than comparable parametric tests.
2. Hypotheses tested with nonparametric methods are less specific than those tested comparably with parametric methods.
3. They do not take advantage of all of the special characteristics of a distribution. Consequently, these methods do not fully utilize the information known about the distribution.

In using nonparametric methods, you should be careful to view them as complementary statistical methods rather than attractive alternatives. With a knowledge of their advantages and disadvantages and some experience, you should be able to determine easily which statistical test is the most appropriate for a given application.

An inherent characteristic of many nonparametric statistics is that they deal with ranks rather than values of the observations. The observations are arranged in an array, and ranks are assigned from 1 to n. Consequently, computations are simple; you deal only with positive integers: $1, 2, 3, \ldots, n$. When working with ranks we often need to compute the sum of the numbers 1 through n, which, we recall from algebra, equals $n(n + 1)/2$. For example, the sum of the first 10 integers is $10(10 + 1)/2 = 55$.

Though there are numerous nonparametric methods, we will limit ourselves to those that correspond to parametric t tests for independent samples, dependent samples, and correlation coefficients. These techniques are the Wilcoxon rank-sum test, the Wilcoxon signed-rank test, the Spearman rank-order correlation coefficient, the Kruskal–Wallis one-way ANOVA, and the Sign Test. We will also present the Fisher's exact test, which is to be used when the χ^2 test would not be valid to use.

4.3   WILCOXON RANK-SUM TEST

The **Wilcoxon rank-sum test** is used to test the null hypothesis that there is no difference in the two population distributions. Based on the ranks from two independent samples, it corresponds to the t test for independent samples, except that no assumptions are necessary as to normality or equality of variances.

To carry out this test with data from Table 14.1, we proceed as follows:

Table 14.1 Wilcoxon Rank-Sum Test for Two Independent Samples: Number of Prenatal-Care Visits for Mothers Bearing Babies of Low and of Normal Birthweight

Mothers Bearing Low-Birthweight Babies			Mothers Bearing Normal-Birthweight Babies		
No.	X (Number of Visits)	R (Rank)	No.	X (Number of Visits)	R (Rank)
1	3	5.5*	1	4	7.5*
2	0	1.5*	2	5	9
3	4	7.5*	3	6	10
4	0	1.5*	4	11	15
5	1	3	5	7	11
6	2	4	6	8	12
7 ($= n_1$)	3	5.5*	7	10	14
			8 ($= n_2$)	9	13
		$W_1 = 28.5$			$W_2 = 91.5$
		$\bar{R}_1 = 4.1$			$\bar{R}_2 = 11.4$

*Two-way tie.

1. Combine the observations from both samples and arrange them in an array from the smallest to the largest.
2. Assign ranks to each of the observations.
3. List the ranks from one sample separately from those of the other.
4. Separately sum the ranks for the first and second samples.

   Given the hypothesis that the average of the ranks is approximately equal for both samples, the test statistic W_1 (the sum of the ranks of the first sample), should not differ significantly from W_e (the expected sum of the ranks). Accordingly, we can show that the expected sum of the ranks for the first sample is

$$W_e = \frac{n_1(n_1 + n_2 + 1)}{2} \tag{14.1}$$

   We have shown that if we obtain W_1's from repeated samples of lists of ranks, the standard error, σ_{w}, is

$$\sigma_w = \sqrt{\frac{n_1 n_2(n_1 + n_2 + 1)}{12}} \tag{14.2}$$

We have further shown that, regardless of the shape of the population distribution, the sampling distribution for the sum of a subset of ranks is approximately normal. Consequently, we have what we need to perform a test of significance regarding the equality of the distributions, namely,

$$Z = \frac{W_1 - W_e}{\sigma_w} = \frac{W_1 - W_e}{\sqrt{n_1 n_2(n_1 + n_2 + 1)/12}} \tag{14.3}$$

Utilizing the data of Table 14.1, we can compute the Z statistic, which compares W_1, the sum of the sample ranks, to W_e, the value that would be expected if the hypothesis were true.

This test assumes that if the first sample has primarily smaller observations than the second sample, then the rank values obtained from the combined sample will be small, giving a small W_1. This implies that the values of the first distribution will be located on the lower end of the combined distribution—which, of course, is contrary to the H_0 that the two distributions are equal.

In attempting to rank the data in Table 14.1, we notice that we have three two-way ties, for zero, three, and four visits. Traditionally, the procedure is to assign the average of the ranks to each tie. For example, the two zeros rank first and second, so we assign them both the average rank of 1.5.

To compute the Z statistic, we will need the expected rank sum. To obtain it, we use equation 14.1:

$$W_e = \frac{n_1(n_1 + n_2 + 1)}{2} = \frac{7(7 + 8 + 1)}{2} = 56 \tag{14.4}$$

To determine whether there is a significant difference between the observed sum of 28.5 obtained from Table 14.1, and the expected value of 56, we use equation 14.3:

$$
\begin{aligned}
Z &= \frac{W_1 - W_e}{\sqrt{n_1 n_2 (n_1 + n_2 + 1)/12}} \\
&= \frac{28.5 - 56}{\sqrt{7(8)(15 + 1)/12}} \\
&= \frac{-27.5}{\sqrt{74.67}} = \frac{-27.5}{8.6} = -3.2
\end{aligned}
$$

From this, we see that the mothers with the low-birthweight infants had a rank sum of 28.5, considerably lower than the expected rank sum of 56. In fact, the observed rank sum falls 3.2 standard errors below the mean of a normal distribution of rank sums. So our conclusion, based on rank sums, is that the mothers bearing low-birthweight infants had a significantly lower number of prenatal visits than the mothers bearing normal-birthweight infants. This conclusion is not surprising, as we can see from comparing the average rankings of the prenatal care visits of the two groups of mothers: $R_1 = 4.1$ versus $R_2 = 11.4$.

We are able to perform this Z test because W is approximately normally distributed. This situation holds if we have at least six cases in each of the groups. Can we perform exact tests if we have smaller sample sizes? Yes. For such methods, with accompanying tables, see an advanced text such as Brown and Hollander (1977).

As mentioned earlier, the rank-sum test parallels the *t* test for two indepen-
dent samples, but it is less powerful. Its power efficiency is greater than 92%,
measured by the performance of repeated rank-sum tests on normally distrib-
uted data.

14.4   WILCOXON SIGNED-RANK TEST

In previous chapters, we also considered the paired *t* test for matched observa-
tions. The counterpart nonparametric test to this is the **Wilcoxon signed-rank
test.** With this test, we assume that we have a series of pairs of dependent ob-
servations. We wish to test the hypothesis that the median of the first sample
equals the median of the second; that is, there is no tendency for the differences
between the outcomes before and after some condition to favor either the before
or the after condition.

The procedure is to obtain the differences (*d*) between individual pairs of ob-
servations. Pairs yielding a difference of zero are eliminated from the computa-
tion; the sample size is reduced accordingly.

To perform the test, we rank the absolute differences by assigning ranks of 1
for the smallest to *n* for the largest. If ties are encountered, they are treated as be-
fore. The signs of the original differences are restored to each rank. We obtain
the sum of the positive ranks, W_1, which serves as the test statistic. If the null hy-
pothesis is true, we would expect to have about an equal mixture of positive
and negative ranks; that is, we would expect the sum of the positive ranks to
equal that of the negative ranks.

Using the data in Table 14.2 on pregnancy and smoking, we see that, because
each pair of observations is on the same woman, we have *dependent* samples;
therefore the Wilcoxon signed-rank test is the appropriate one to perform. The
column denoted by *d* represents differences (before and after pregnancy); the
column labeled r_d is the rank by size of the absolute difference. Rank 1 is as-
signed to the smallest and *n* (here, 10) to the largest. Now we can obtain W_1 and
W_2, the sums, respectively, of the positive and negative ranks. Recall that the
sum of all ranks is $n(n + 1)/2$. Under the null hypothesis we assume that the
sum of the ranks of the positive *d*'s is equal to the sum of the ranks of the nega-
tive *d*'s; that is, each will be half of the total sum of the ranks, or, algebraically,
the expected sum of the ranks will be

$$W_e = \left(\frac{1}{2}\right)\frac{n(n + 1)}{2} \tag{14.5}$$

which, for the data of Table 14.2, is $10(11)/4 = 27.5$. The test statistic is the
smaller of the sums, namely, $W_1 = 7$.

Table 14.2 Wilcoxon Signed-Rank Test: Number of Cigarettes Usually Smoked per Day, Before and After Pregnancy

| Subject | Number of Cigarettes Smoked per Day | | $d = x_a - x_b$ | $|d|$ | r_d |
	x_b: Before Pregnancy	x_a: After Pregnancy			
1	8	5	−3	3	3(−)
2	13	15	+2	2	2(+)
3	24	11	−13	13	9(−)
4	15	19	+4	4	4(+)
5	7	0	−7	7	7(−)
6	11	12	+1	1	1(+)
7	20	15	−5	5	5(−)
8	22	0	−22	22	10(−)
9	6	0	−6	6	6(−)
10	15	6	−9	9	8(−)
11	20	20	0	—	—

$$\Sigma r_d = \frac{n(n+1)}{2} = \frac{10(11)}{2} = 55 \qquad \Sigma r_{d(+)} = W_1 = 7$$

$$W_e = \frac{\Sigma r_d}{2} = \frac{55}{2} = 27.5 \qquad \Sigma r_{d(-)} = W_2 = 48$$

Because W_1 is approximately normally distributed with a mean of W_e and a standard deviation of σ_w, we are able to perform a Z test for the difference between the sums of the matched ranks by using the following equation:

$$Z = \frac{W_1 - W_e}{\sigma_w}$$

$$= \frac{W_1 - W_e}{\sqrt{(2n + 1)W_e/6}}$$

$$= \frac{7 - 27.5}{\sqrt{[2(10) + 1]27.5/6}} \qquad (14.6)$$

$$= \frac{-20.5}{\sqrt{96.25}}$$

This result indicates that the difference between the observed and expected rank sums is significant ($p < .05$). Thus, it leads us to reject the H_0 that there is no difference between smoking status before and after pregnancy. The implication: There is a significant reduction in the smoking habit consequent to pregnancy.

The Wilcoxon signed-rank test has a power efficiency of 92% as compared with paired t tests, which satisfy the assumption of normality. Note that this

technique is somewhat less sensitive than the parametric one in that the ranks do not directly describe the amount of reduction in smoking.

The assumption of normality for the sum of the signed-rank test is appropriate, providing you have at least eight pairs. For a smaller sample size, you will need an exact test. Tables for such a test are available in more advanced textbooks, such as Brown and Hollander (1977), which also includes confidence intervals for the Wilcoxon tests.

A natural question arises here: Does a nonparametric procedure exist for making comparisons of more than two groups? That is, is there a parallel nonparametric ANOVA test? There is; it is called the **Kruskal–Wallis test.** For a discussion, see a text such as Steel and Torrie (1980).

14.5   KRUSKAL–WALLIS ONE-WAY ANOVA BY RANKS

The Kruskal–Wallis test is the nonparametric equivalent of the one-way ANOVA. This technique is an alternative to the one-way ANOVA when you have three or more groups, the groups are independent, and the populations from which the samples are selected are not normally distributed or the samples do not have equal variances. It can also be used when you have ordered outcomes—that is, ordinal data rather than the interval or ratio data necessary to use an ANOVA. For example, suppose the rows represent three or more pain relievers and the columns represent distinct, ordered responses. These responses might be no relief, mild relief, moderate relief, strong relief, and complete relief (Mehta, 1994). The example used in this chapter will start with ratio data and the assumption that the one-way ANOVA is not the appropriate procedure because of one of the reasons just described.

To use the Kruskal–Wallis technique you combine the observations of the various groups. After arranging them in order of magnitude from lowest to highest you then assign ranks to each of the observations and replace them in each of the groups. What you have just done is convert the original ratio data into ordinal or ranked data. If you started with ordinal data, this conversion would not be necessary.

Next the ranks are summed in each of the groups and the test statistic H is computed. The rank assigned to observations in each of the K groups are added separately to give K rank sums.

The test statistic is computed using

$$H = \frac{12}{N(N + 1)} \sum \frac{R_j^2}{n_j} - N(N + 1)$$

$$= 12 \sum \frac{R_j^2}{n_j} - N(N + 1)$$

(14.5)

In this quotation

k = the number of groups

n_j = the number of observations in the jth group

N = the number of observations in all groups combined

R_j = the sum of the ranks in the jth group

Let us look at performance scores of three different types of teachers.

	Teacher Type	
A	B	C
96	8	115
128	124	149
83	132	166
61	135	147
101	109	—

The table of corresponding ranks is shown here.

	Teacher Type	
A	B	C
4	2	7
9	8	13
3	10	14
1	11	12
5	6	—
$R_1 = 22$	$R_2 = 37$	$R_3 = 46$

Using an example from Siegal (1956) 3, we can now calculate statistic H.

$$H = \frac{12}{14(14 + 1)} \sum^{k} \frac{R_j^2}{n_j} - 3(N + 1) = \frac{12}{14(15)}\left(\frac{22^2}{5} + \frac{37^2}{5} + \frac{46^2}{4}\right) - 3(14 + 1)$$

$$= 6.4$$

When we refer to Appendix F, it shows that when the n_j's are 5, 5, and 4, $H \geq 6.4$ has probability of occurrence under the null hypothesis of ($p < .049$). However, because the probability is smaller than $\alpha = .05$, our decision in this study is to reject H_0, and we conclude that the three groups of educators differ in their scores.

Tied Observations

When two or more scores are tied, each score is given the mean of ranks for which it is tied. Because H is somewhat influenced by ties, one may wish to correct for ties in computing H. To correct for the effect of ties, H is computed from the formula on the previous page and divided by

$$1 - \frac{\Sigma T}{N^3 - N}$$

where T is the number of tied observations in a tied group of scores.

The effect of correcting for ties is to increase the value of H and thus make the result more significant than it would have been if H remained uncorrected. In most cases the effect of correction is negligible. With even 25% of observations involved in ties, the probability associated with an H computed without the correction for ties is rarely changed by more than 10% when the correction of ties is made. (See Mehta, 1994.)

14.6   THE SIGN TEST

The **sign test** is one of the simplest of statistical tests. It focuses on the median rather than the mean as a measure of central tendency. The only assumption made in performing this test is that the variables come from a continuous distribution.

It is called the "sign test" because we use pluses and minuses as the new data in performing the calculations. We illustrate its use with a simple sample and a paired sample. The sign test is useful when we are not able to use the t test because the assumption of normality has been violated.

Single Sample

In the case of a single sample, we wish to test the H_0 that the sample median is equal to population median m. To do this, we assign ($+$) to observations that fall above the population median and ($-$) to those that fall below the population median. A tie is given a zero and is not counted. If the H_0 is true—that the medians are the same—we expect an equal number: 50% pluses and 50% minuses.

We can use the binomial distribution to determine if the number of positive signs deviates significantly from some expected number. Instead of using the binomial equations, however, we can use the table in Appendix D. This table shows the probability of having the observed number of pluses when we expect 50%.

■ **EXAMPLE 1**

In an anesthetic used for major surgery, the median number of hours it takes for the anesthesia to wear off is 7. A new agent has been suggested that supposedly

provides relief much sooner. In a series of 12 surgeries using the new anesthetic, the following times for recovery were observed:

Recovery time:	4	4	5	5	5	6	6	6	7	7	8	9
Sign:	−	−	−	−	−	−	−	−	0	0	+	+

H_0: The median recovery time for the new anesthetic is 7 hours.

Because the suggestion is made that the new anesthetic is better, we have a one-tailed test. We can see that eight outcomes are less than the median of the standard anesthetic and two are more. We exclude the two with a score of 7 hours because they are equal to the median of 7. Thus, we observe two pluses when we expected 5 pluses under the H_0.

To determine the probability that the two pluses occur randomly, we can use the binomial distribution formula or the table of critical values for the sign test shown in Appendix D. From this table we can see, for $n = 10$ and for an $\alpha = .05$ one-sided test, that $q = 1$ or 9. Therefore, for this sample to have a significantly better recovery time than 7 hours, it will require nine minuses or one plus. We observed eight minuses, so we cannot reject H_0; that is, we find no statistically significant difference between the recovery time from the sample and that of the population median. ∎

Paired Samples

The sign test is also suitable for experiments with paired data such as before or after, or treatment and control. In this case, we need to satisfy only one assumption—that the different pairs are independent; that is, only the direction of change in each pair is recorded as a plus or minus sign. We expect an equal number of pluses or minuses if there is no treatment effect. The H_0 tested by the paired samples sign test is that the median of the observations listed first is the same as that of the observations listed second in each pair.

∎ EXAMPLE 2

Ten blood samples were sent to two labs for cholesterol determinations. The results from the two labs are as follows:

Serum Cholesterol Determinations Obtained from Two Labs on the Same Samples

Patient	1	2	3	4	5	6	7	8	9	10
Lab A	296	268	244	272	240	244	282	254	244	262
Lab B	318	287	260	279	245	249	294	271	262	285
Sign of difference	−	−	−	−	−	−	−	−	−	−

H_0: The median serum cholesterol determination of both labs is equal.

We can see that all 10 observations are minuses. What are the chances of obtaining such a result by chance? From the table in Appendix D we can see that for $n = 10$ the critical value of q is either 1 or 9 for an $\alpha = .05$ two-sided test. Consequently, because our result was equal to 10, we conclude that there is a significant difference in the way the two labs determine cholesterol levels. ∎

14.7   SPEARMAN RANK-ORDER CORRELATION COEFFICIENT

In Chapter 13, we discussed in detail the Pearson correlation coefficient, which describes the association between measurement variables x and y. In this section, we discuss an association between two ranked variables. With the **Spearman rank-order correlation coefficient,** we obtain perfect correlation (± 1) if the ranks for variables x and y are equal for each individual. Conversely, lack of association is measured by examining the differences in the ordered ranks, $d_i = x_i - y_i$. The Spearman rank-order correlation coefficient, r_s (the s is for Spearman), can be derived from the Pearson correlation coefficient r. The equation is

$$r_s = 1 - \frac{6\Sigma d_i^2}{n(n^2 - 1)} \tag{14.7}$$

where d_i is the difference between the paired ranks and n is the number of pairs. Like the Pearson correlation coefficient, the Spearman rank-order correlation coefficient may take on values from -1 to $+1$. Values close to ± 1 indicate a high correlation; values close to zero indicate a lack of association. The minus or plus signs indicate whether the correlation coefficient is negative or positive.

To illustrate the use of the Spearman rank-order correlation coefficient, let us consider a situation that is all too familiar to any college student. The work of 12 students is observed independently by two faculty evaluators, who rank their performance from 1 to 12 (Table 14.3). As before, ties in rank are handled by averaging the ranks. Note that observer C had a three-way tie for first place. The x and y columns of Table 14.3 are the ranks, the d_i column is the difference between the ranks, and the final column is d_i^2. Using equation 14.7, we obtain

$$r_s = 1 - \frac{6\Sigma d_i^2}{n(n^2 - 1)}$$

$$= 1 - \frac{6(55.50)}{12(144 - 1)} = .81$$

To determine whether this coefficient differs significantly from zero, we need to assume that x and y represent randomly selected and independent pairs of ranks. We can use the same test procedure as for the Pearson r. It provides a good approximation if the sample size is at least 10. The equation for the test statistic is

$$t = \frac{r_s \sqrt{n-2}}{\sqrt{1-r_s^2}} \tag{14.8}$$

with $n-2$ df. Using the data from Table 14.3, we find that

$$t = \frac{.81\sqrt{10}}{\sqrt{1-.66}} = \frac{(.81)(3.16)}{.58} = 4.41$$

Because the computed t of 4.41 is greater than the critical $t_{.95}$ of 2.23 for 10 df, we reject H_0 and conclude that the correlation differs significantly from zero.

Whenever you are able to meet the assumptions for computing a Pearson r, use it. It is preferable to the Spearman r_s because the power of the latter is not as great as that of r. For samples having 10 or fewer observations, see advanced textbooks such as Dixon and Massey (1969) and Brown and Hollander (1977), which also give tabulations for critical values of r_s.

Table 14.3 Ranking of Students' Performance by Two Independent Observers

Student No.	Observer B: Rank Order (x)	Observer C: Rank Order (y)	$d_i = x_i - y_i$	$d_i^2 = (x_i - y_i)^2$
1	2.5*	5	-2.5	6.25
2	2.5*	2†	0.5	0.25
3	9	8	1.0	1.00
4	5.5*	7	-1.5	2.25
5	12	12	0	0
6	7.5*	11	-3.5	12.25
7	1	2†	-1.0	1.00
8	10	6	4.0	16.00
9	4	2†	2.0	4.00
10	5.5*	4	1.5	2.25
11	7.5*	10	-2.5	6.25
12	11	9	2.0	4.00
				$\Sigma d_i^2 = 55.50$

*Two-way tie.
†Three-way tie.

14.8   FISHER'S EXACT TEST

The chi-square test described in Chapter 12 has a limitation. It is not appropriate for a situation in which the sample size is small, yielding small expected frequencies. There should be no expected frequencies less than 1, and not more than 20% of the expected frequencies are to be less than 5. For a situation like this, we should consider using **Fisher's exact test,** which computes directly the probability of observing a particular set of frequencies in a 2 × 2 table. It is calculated using the following formula:

$$P = \frac{(a+b)!\,(c+d)!\,(a+c)!\,(b+d)!}{N!\,a!\,b!\,c!\,d!} \qquad (14.9)$$

where a, b, c, and d are the frequencies of a 2 × 2 table and N is the sample size.

■ EXAMPLE 3

An infant heart transplant surgeon had nine infant patients who needed a heart transplant. Only five suitable donors were identified. A follow-up of the nine patients was done a year later to see if there was a difference in the survival rates of those with and without heart transplants. The following results were found:

Heart Transplant Candidates

		Alive 12 Months Later?		
		Yes	No	Total
Surgery	Yes	$a = 4$	$b = 1$	5
Performed	No	$c = 1$	$d = 3$	4
	Total	5	4	9

The probability of observing this particular set of frequencies is

$$P = \frac{5!\,4!\,4!\,5!}{9!\,4!\,1!\,1!\,3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 4}{6 \cdot 7 \cdot 8 \cdot 9} = \frac{20}{126} = 0.159$$

However, to compute the P value, we need to find the probability of obtaining this or a more extreme result while keeping the marginal totals in the table fixed. A more extreme result would be, for example, if all of the infants without a heart transplant were dead a year later. To do this, we reduce by 1 the smallest frequency that is greater than zero while holding the marginal totals constant in the table on heart transplant candidates. This gives the following 2 × 2 table:

5	0	5
0	4	4
5	4	9

The probability of obtaining this set of frequencies is

$$P = \frac{5! \, 4! \, 4! \, 5!}{9! \, 5! \, 0! \, 0! \, 4!} = \frac{5! \, 4!}{9!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 7 \cdot 8 \cdot 9} = \frac{1}{126} = .008$$

Thus, the probability of observing this particular frequency of successful transplants or a more extreme frequency is $0.159 + 0.008 = 0.167$. This P value is for a one-tail test. An estimate of the P value for a two-tail test is obtained by multiplying the value by 2: $2 \times 0.167 = 0.334$. Based on this outcome, we would fail to reject the H_0 that there is no difference in the survival rate between infants with or without a heart transplant. Although this result may be difficult to accept, it is the best we can do with such a small sample. There has been some controversy as to whether it is appropriate to use Fisher's exact test in the health sciences because the model requires that the marginal totals in the 2×2 table be fixed—and they seldom are in actual health science settings. Nevertheless, some statisticians use this test anyway because the test results tend to give conservative values of P; that is, the true P value is actually less than the computed one. ■

Conclusion

There are nonparametric methods that correspond to such parametric methods as the t test, paired t test, and correlation coefficient. The primary advantage of these methods is that they do not involve such restrictive assumptions as those of normality and homogeneity of variance. Their major disadvantage is that they are less efficient than the corresponding parametric methods of the five methods described here—the Wilcoxon rank-sum test, the Wilcoxon signed-rank test, the sign test, the Spearman rank-order correlation coefficient, and Fisher's exact test. These are the nonparametric methods used most frequently in the health sciences.

Vocabulary List

Fisher's exact test	parametric methods	Wilcoxon rank-sum test
Kruskal–Wallis test	sign test	Wilcoxon signed-rank
nonparametric methods	Spearman rank-order	test
(distribution-free	correlation coefficient	
methods)		

Exercises

14.1 To learn if babies who were breast-fed had a better dental record than those who were not, 13 children were picked at random to see at what age they acquired their first cavities. The results were as follows:

Subject	Breast-Fed— Yes/No	Age at First Cavity
1	No	9
2	No	10
3	Yes	14
4	No	8
5	Yes	15
6	No	6
7	No	10
8	Yes	12
9	No	12
10	Yes	13
11	No	6
12	No	20
13	Yes	19

a. State the null hypothesis.
b. State the alternative hypothesis.
c. Do a Wilcoxon rank-sum test.

14.2 Refer to Table 2.2. Compute a Wilcoxon rank-sum test to determine whether there is a significant difference in diastolic blood pressure between
a. vegetarian males and nonvegetarian males
b. vegetarian males and vegetarian females

14.3 Two communities are to be compared to see which has a better dental record. Town A has fluoride in the water; Town B does not. Ten persons are randomly picked from each town and their dental cavities are counted and reported. The data are as follows:

	Person									
	1	2	3	4	5	6	7	8	9	10
Town A	0	1	3	1	1	2	1	2	3	1
Town B	3	2	2	3	4	3	2	3	4	3

a. State the null hypothesis.
b. State the alternative hypothesis.
c. Do a Wilcoxon rank-sum test.

14.4 There are two methods of counting heartbeats: (1) by counting the pulse at the wrist and (2) by counting the pulse on the neck. An investigator wishes to know the degree of correlation between the two methods. The data are as follows:

					Person					
	1	2	3	4	5	6	7	8	9	10
Neck pulse	73	99	77	63	50	80	83	73	66	82
Wrist pulse	74	103	77	61	51	81	82	74	66	83

a. State the null hypothesis.
b. State the alternative hypothesis.
c. Do a Spearman rank-order correlation coefficient. (*Hint:* Rank the neck pulse from highest to lowest; do the same for the wrist pulse.)

14.5 Two health inspectors rate 11 hospitals on cleanliness, as shown in the tabulation that follows. Determine if their rankings are comparable.

						Hospital					
	1	2	3	4	5	6	7	8	9	10	11
Inspector 1	2	3	2	3	1	4	5	3	1	3	4
Inspector 2	1	3	3	2	2	5	4	2	1	4	3

a. State the null hypothesis.
b. State the alternative hypothesis.
c. Perform the appropriate test.

14.6 a. What is meant by nonparametric methods?
b. What are their advantages?
c. What are their disadvantages?

14.7 Describe the conditions that call for using each of the following tests:
a. Wilcoxon rank-sum test
b. Wilcoxon signed-rank test
c. Spearman rank-order correlation coefficient

14.8 A group of 11 hypertensive individuals determined to find out if they could lower their systolic blood pressure through a systematic physical fitness program. They observed their blood pressure before they began their program and then again six months later. The following results were found:

					Systolic Blood Pressure						
Case	1	2	3	4	5	6	7	8	9	10	11
Before	156	130	142	155	174	140	148	152	156	136	126
After	148	124	135	146	169	145	140	156	161	133	123

a. State the H_0 and H_1.
b. Perform the Wilcoxon signed-rank test at the $\alpha = .05$ level.

14.9 a. Calculate a Spearman rank-order correlation coefficient on the data of Exercise 14.8.

b. State the H_0 and H_1 and perform a test of significance at the $\alpha = .01$ level.

14.10 For the data in Exercise 9.14, determine whether the experimental condition increases the number of heartbeats per minute.

a. State the H_0 and H_1.

b. Perform the appropriate test of significance at the $\alpha = .05$ level.

14.11 An investigator observed the following response to two different dental treatments:

	Treatment A	Treatment B	Total
Favorable	4	2	6
Not favorable	1	4	5
Total	5	6	11

a. Determine whether the differences in response rates between treatments A and B are significant at $\alpha = .05$.

b. What was the H_0 that you tested in (a)?

14.12 a. Using the data of Exercise 14.8, perform a sign test. Indicate the H_0 you are testing and whether or not you would reject it at the $\alpha = .05$ level.

b. Using the data of Exercise 14.4, perform a sign test. Indicate the H_0 you are testing and whether or not you would reject it at the $\alpha = .05$ level.

14.13 A study investigated dietary differences between low income African-American women and low income white women. One dietary practice examined was the daily serving of meats (1 serving = 3 oz. edible portion of meat). Based on the following table, is there any reason to believe that there are differences between low income African-American women and low income white women, with respect to their consumption of meats?

Meats													
Number of servings													
African American	0	0	1	1	1	2	2	3	3	3	3	3	5	6
White	0	0	0	1	2	2	2	2	3	3	4

NOTE: This data was extrapolated and based on Cox (1994).

a. State the null hypothesis.

b. State the alternative hypothesis.

c. Do a Wilcoxon rank-sum test.

14.14 Exercise 14.13 is based on an actual study conducted by Ruby Cox. In her study, there were 115 African-American women and 95 white women sampled. When she used the Wilcoxon procedure, she found a significant difference in the daily meat consumption of the two groups of women (African-American women consumed more). If you calculated your statistics correctly for Exercise 14.13, you

did not find a significant difference. This illustrates a weakness of the Wilcoxon rank-sum test. What is that weakness?

14.15 Nurses are often expected to subjectively evaluate a patient's comfort level. A nurse researcher wanted to determine if the subjective ranking done by 2 nurses (working with the same patients) of comfort levels of 15 patients were similar. Based on the following data, is there a relationship between the rankings of the two nurses?

Patient	Comfort Rank	
	Nurse 1	Nurse 2
A	2	1
B	4	3
C	12	14
D	1	2
E	15	11
F	8	8
G	3	6
H	6	4
I	11	13
J	9	10
K	5	5
L	14	15
M	10	9
O	7	7
P	13	12

15 Vital Statistics and Demographic Methods

Chapter Outline

15.1 Introduction
Points out the importance of vital statistics and demographics

15.2 Sources of Vital Statistics and Demographic Data
Discusses three sources of data—census data, registration of births and deaths, and morbidity data—as the building blocks for computing vital rates, ratios, and proportions

15.3 Vital Statistics Rates, Ratios, and Proportions
Introduces the concepts of rates, ratios, and proportions within the context of vital statistics

15.4 Measures of Mortality
Presents a variety of measures, each being a means of measuring the frequency of deaths in a community

15.5 Measures of Fertility
Shows two key methods for quantifying fertility that are indispensable in making population estimates

15.6 Measures of Morbidity
Describes three of the many measures of illness that exist

15.7 Adjustment of Rates
Explains how to make reasonable comparisons between noncomparable populations

Learning Objectives

After studying this chapter, you should be able to

1. Distinguish among
   a. rates, ratios, and proportions
   b. measures of morbidity, mortality, and fertility
2. Compute and understand the meaning of various vital measures
3. State the reasons why measures are adjusted
4. Compute an adjusted rate by the direct method

i.1   INTRODUCTION

Decision making in the health sciences, especially public health, is continually becoming more quantitative. Demographic data and **vital statistics** have emerged as indispensable tools for researchers, epidemiologists, health planners, and other health professionals. To determine the health status of a community, to decide how best to provide a health service, to plan a public health program, or to evaluate a program's effectiveness, it is essential to use these tools knowledgeably.

Demographic variables describe a population's characteristics—for instance, its size and how that changes over time; its composition by age, sex, income, occupation, and utilization of health services; its geographic location and density. Once you possess demographic data and information about **vital events** (births, deaths, marriages, and divorces), you can tackle a remarkable variety of problems regarding a community's status at a particular time or its trends over a period. Together with measures of illness and disease, demographic data are invaluable in program planning and disease control. Such data also go a long way toward providing research clues as to the often-unexpected associations between a population's health practices and its disease experience.

A wide array of methodological tools is available to deal with such data. In this chapter, we consider *vital rates, ratios, proportions, measures of fertility* and *morbidity,* and *adjustment of rates.* But first, we discuss some of the sources of demographic data and vital statistics.

.2   SOURCES OF VITAL STATISTICS
AND DEMOGRAPHIC DATA

The three main sources of demographic data, vital statistics, and morbidity data are the census, registration of vital events, and morbidity surveys. These are usually given for defined populations such as cities, states, and other political areas. Traditionally, hospital and clinic data have also been major sources of morbidity data. Although they are useful, however, the populations of reference may be difficult to define.

The Census

The United States has conducted **decennial census** of the population since 1790. In a census, each household and resident is enumerated. Information obtained on each person includes his or her sex, age, race, marital status, place of residence, and relationship to or position as the head of household. A systematic sample of households then provides more information, such as income, housing, number of children born, education, employment status, means of

transportation to work, and occupation. Census tables are published for the entire United States, for each state, for **Metropolitan Statistical Areas (MSAs),** for counties, and for cities, neighborhoods (census tracts), and city blocks. The MSAs are urbanized areas. An area qualifies as an MSA if it has one or more cities of at least 50,000 residents and there is a social and economic integration of the cities with the surrounding rural areas. In the 1990 census, there were 332 MSAs (including 5 in Puerto Rico).

Census results are published in the *Decennial Census of the United States* about two years after the census is taken. They are also made available on magnetic tape for computerized analysis. A good deal of census information is summarized annually in the *Statistical Abstract of the United States.* The importance of census data is universally recognized. More than four-fifths of the world's population is counted in some kind of census at more or less regular intervals.

Annual Registration of Vital Events

As noted earlier, vital events are births, deaths, marriages, and divorces. In the United States, state laws require that all vital events be registered. Registration is now quite complete and reliable. Birth certificates serve as proof of citizenship, age, birthplace, and parentage; death certificates are required as burial documents and in the settlement of estates and insurance claims. In the United States, **death registration** began in Massachusetts in 1857, was extended to 10 states, the District of Columbia, and several other cities by 1900, and has been nationwide since 1933. **Birth registration** began in 1915, encompassing 10 states and the District of Columbia. By 1933, all states had been admitted to the nationwide birth and death registration system. A great deal of information is recorded on birth and death certificates. Some of the key elements are as follows:

Birth certificate	Death certificate
Name	Name
Sex	Date and time of death
Date and time of birth	Race
Weight and length at birth	Age
Race of parents	Place of birth
Age of parents	Names of decedent's parents
Birth order	Name and address of survivor
Occupation of father	(or informant)
Place of birth	Marital status
Residence of mother	Occupation

Birth certificate	Death certificate
Physician's (or attendant's) certification	Place of residence
	Cause(s) of death
	Place of death
	Burial data
	If death due to injury: accident, suicide, or homicide
	Physician's (or coroner's) certification

The National Center for Health Statistics collects a systematic sample of 10% of the births and deaths in each state. From this, it publishes the monthly *Vital Statistics Report.* Annually, it issues the four-volume set *Vital Statistics of the United States,* which includes many detailed tables on vital events for all sorts of demographic characteristics and for major geographical subdivisions. Data on marriages and divorces are similarly collected and published in a separate volume of *Vital Statistics of the United States.*

The federal government has been instrumental in getting the various states to adopt standard birth and death certificates. This enables researchers to collect more standardized information. All states now compile computerized death certificate data, or "death tapes," which are computer-readable extracts of the most important data appearing on death certificates. Since 1979, the National Center for Health Statistics has prepared the National Death Index, a nationwide, computerized index of death records compiled from tapes submitted by the vital statistics offices of each state. These tapes contain a standard set of identifying data for each decedent. The index (National Center for Health Statistics, 1981) permits researchers to determine if persons in their studies have died; for each such case, the death certificate number is available, along with the identity of the state where the death occurred and the date of death. Given these **mortality data,** the researcher can order a copy of the death certificate from the state's vital statistics office.

One of the tasks performed by the National Center for Health Statistics is to classify deaths into various numerical categories. This very complex task is performed by nosologists who use the two current volumes on how to classify a particular cause of death (COD). This classification is then used to tabulate data according to various codes.

Morbidity Surveys

Morbidity data (i.e., data on the prevalence of disease) are far more difficult to gather and interpret than are mortality data. Whereas death registration is now

estimated to be 99% complete, cases of communicable disease are all too often underreported.

Reporting of communicable diseases is a time-honored, if flawed, method of gathering morbidity data. In 1876, Massachusetts tried voluntary case reporting; the first compulsory reporting began in Michigan in 1883 (Winslow et al., 1952). But even now, a century later, there are wide gaps in the data. California, for example, has 52 reportable diseases; other states have fewer. Various surveys have concluded that the more serious diseases are well reported. But whereas virtually every case of cholera, plague, yellow fever, rabies, and paralytic polio is promptly brought to the attention of health authorities, the common childhood diseases are notoriously underreported.

Each local health department tallies the number of cases of reportable communicable disease within its area and forwards its count to the state health department, where a cumulative total is made and sent to the Centers for Disease Control in Atlanta for publication in *Morbidity and Mortality Weekly Reports (MMWR)*.

Because of this chronic underreporting, a number of novel systems have been developed to make better estimates of morbidity data. In the following partial list of these systems, note that many of them go far beyond communicable-disease reporting to include data on noninfectious, occupational, and chronic diseases.

1. Reportable diseases
2. National Health Survey
3. Hospital records data
4. Industrial hygiene records
5. School nurse records
6. Medical care subgroups (most often: prepaid medical plans)
7. Chronic-disease registries (most often: tumor registries)
8. Insurance industry data

The **National Health Survey** is worthy of special note. Originated by an Act of Congress in 1956, it provides for an annual nationwide survey of a representative sample of 40,000 persons. A number of subprograms are included, the most notable of which are the National Health Interview Survey, National Health and Nutrition Examination Survey (HANES), National Hospital Discharge Survey, National Ambulatory Medical Care Survey, and National Nursing Home Survey. The results are published in *Vital and Health Statistics*, sometimes referred to (from its colorful covers) as the "rainbow series." Published results encompass a vast spectrum of medical care data, including incidence or prevalence rates for many diseases, length of hospital stays, hospitalizations by cause, number of days of disability, and patterns of ambulatory care service.

Hospital and clinic records are a fair source of morbidity data. However, ex-

cept for prepaid medical plans, the population served by a hospital is hard to define. The Professional Activity Study of Battle Creek, Michigan, provides a uniform reporting system that is used by over 2000 hospitals nationwide. Researchers use this system to make morbidity estimates for population studies. Hospital administrators find this and other resources to be invaluable for planning strategies of health care delivery.

Chronic-disease registries are rapidly taking on a major role in the understanding of morbidity data. Most such registries are cancer-oriented (and are therefore termed cancer, or tumor, registries), although some are specialized for such diseases as cardiovascular disease, tuberculosis, diabetes, and psychiatric disease. A cancer registry is defined as a "facility for the collection, storage, analysis, and interpretation of data on persons with cancer." Some such registries are hospital-based; that is, they work within the walls of a hospital or group of hospitals. Others are population-based, in that they serve a population of defined composition and size. Among the best-known of the latter are the tumor registries of Connecticut and Iowa, each serving the entire state (Muir and Nectoux, 1977).

Although we have focused on data for the United States, similar data are available for most of the developed world. They may be found in the annual *Demographic Yearbook* (United Nations, 1990).

5.3   VITAL STATISTICS RATES, RATIOS, AND PROPORTIONS

The field of vital statistics makes some special applications of rates, ratios, and proportions. A **rate** is an expression of the form

$$\left[\frac{a}{(a + b)t} \right] c \tag{15.1}$$

where

a = the number of persons experiencing a particular event during a given period

$a + b$ = the number of persons who are at risk of experiencing the particular event during the same period

t = the total time at risk

c = a multiplier, such as 100, 1000, 10,000, or 100,000

The purpose of the multiplier, also referred to as the **base**, is to avoid the inconvenience of working with minute decimal fractions; it also helps users to comprehend the meaning of a given rate. We usually choose c to give a rate that is in the tens or hundreds.

Three kinds of rates are commonly used in vital statistics: crude, specific, and adjusted rates. **Crude rates** are computed for an entire population. They disregard differences that usually exist by age, sex, race, or some category of disease. **Specific rates** consider the differences among subgroups and are computed by age, race, sex, or other variables. **Adjusted** (or **standardized**) **rates** are used to make valid summary comparisons between two or more groups possessing different age (or other) distributions.

A **ratio** is a computation of the form

$$\left(\frac{a}{d}\right)c \tag{15.2}$$

where a and c are defined as for rates, and d is the number of individuals experiencing some event different from event a during the same period. Quite commonly used is the sex ratio; by convention, it places males in the numerator and females in the denominator. A ratio of 1.0 would describe a population with an equal number of males and females.

A **proportion** is an expression of the form

$$\left(\frac{a}{a + b}\right)c \tag{15.3}$$

where a, $a + b$, and c are defined as for rates.

15.4   MEASURES OF MORTALITY

A wide variety of rates, ratios, and proportions are based on numbers of deaths. Each rate is a measure of the relative frequency of deaths that occurred in a given population over a specific period. If we know the population and **time at risk,** we can compute a mortality rate. Unfortunately, these figures are sometimes difficult to obtain. A convention is used to define population size: the population at midyear (July 1). The figure obtained serves as a reasonable estimate of the **population at risk** ($a + b$) over the time (t) of one year. If this convention cannot be met, the calculation should preferably be termed a "proportion" rather than a "rate."

In the health sciences, the fine distinctions among rates, ratios, and proportions are often ignored. Consequently, you may find that some sources erroneously term certain ratios as "rates"; the most common of these are starred (*) in the discussion that follows. Some proportions are similarly misnamed "rates"; these also are starred.

Annual Crude Death Rate

The annual crude death rate is defined as the number of deaths in a calendar year, divided by the population on July 1 of that year, the quotient being multiplied by 1000.

■ EXAMPLE 1

California, 1987—population: 27,663,000; deaths: 210,171.

$$\text{Crude death rate} = \frac{210,171}{27,663,000} \times 1000$$

$$= 7.6 \text{ deaths per 1000 population per year} ■$$

The annual crude death rate is universally used. It is indeed crude—a generalized indicator of the health of a population. In our example, the rate of 7.6 deaths per 1000 is a bit less than the overall U.S. death rate of 8.7. But it is often incautious to make such a comparison, especially when the two populations are known to differ on important characteristics such as age, race, or sex. More appropriate comparisons are made by use of adjusted rates. The process of adjustment is a bit involved; we deal with it later. In the meantime, there is another way of making fair comparisons between groups—by use of *specific* rates. Death rates may be specific for age, for sex, or for some particular cause of death.

Age-Specific Death Rate

The age-specific death rate is defined as the number of deaths in a specific age group in a calendar year, divided by the population of the same age group on July 1 of that year, the quotient being multiplied by 1000.

■ EXAMPLE 2

United States, 1987—age group: 25–34 years; population: 43,513,000; deaths: 57,701.

$$\text{Age-specific death rate} = \frac{57,701}{43,513,000} \times 1000$$

$$= 1.3 \text{ deaths per 1000 population per year for age group 25–34} ■$$

Cause-Specific Death Rate

The cause-specific death rate is defined as the number of deaths assigned to a specific cause in a calendar year, divided by the population on July 1 of that year, the quotient being multiplied by 100,000.

■ EXAMPLE 3

United States, 1987—cause: accidents; population: 243,827,000; deaths: 94,840.

$$\text{Cause-specific death rate} = \frac{94,840}{243,827,000} \times 100,000$$

$$= 39.0 \text{ accidental deaths per 100,000 population per year } ■$$

Cause–Race-Specific Death Rate

The cause–race-specific death rate is one of many possible examples of how the idea of specific death rates may be extended simultaneously to cover two characteristics.

■ EXAMPLE 4

United States, 1987—white male population, 100,589,000; nonwhite male population, 17,942,000.

The full data for this example are given in Table 15.1. Note the difference in the death rate between the two racial groups. But the underlying explanation for the difference may be something other than race. What other factor might explain the difference? ■

Table 15.1 Cause–Race-Specific Death Rate, United States, 1987

	White Males	Nonwhite Males
Population	100,589,000	17,942,000
Deaths assigned to accidents	53,936	10,880
Cause–race-specific death rate per 100,000	53.6	60.6

*Proportional Mortality Ratio

Proportional mortality ratio is defined as the number of deaths assigned to a specific cause in a calendar year, divided by the total number of deaths in that year, the quotient being multiplied by 100.

■ EXAMPLE 5

United States, 1987—total deaths from all causes: 2,123,000; deaths assigned to malignant neoplasms: 476,927.

$$\text{Proportional mortality ratio} = \frac{476,927}{2,123,000} \times 100$$

$$= 22.5\% \text{ of total deaths per year from malignant neoplasms } ■$$

■ **EXAMPLE 6**

United States, 1987—persons 15–24 years old: 38,481,000; persons age 65 or over: 29,835,000.

From the full data for this example in Table 15.2, you can see that this proportion is useful as a measure of the relative importance of a specific cause of death. But though it is quite simple to compute, it should be used with caution because it is quite easy to misinterpret. For instance, the proportional mortality for accidental death here is much greater for young adults than for elderly persons. Nevertheless, the death rate from accidents is higher for the elderly. This apparent dilemma disappears when you realize the numerical impact of the large number of deaths from all causes among the elderly.

Table 15.2 Cause-Specific Death Rate, United States, 1987

	Persons Ages 15–24	Persons Age 65 and Over
Population	38,481,000	29,835,000
Deaths—all causes		
Number	38,023	1,509,686
Death rate per 100,000	98.8	5,060.1
Deaths—accidental causes		
Number	18,695	25,838
Death rate per 100,000	48.6	86.2
Proportional mortality—accidental causes (%)	49.2%	1.7%

Proportional mortality is particularly useful in occupational studies as a measure of the relative importance of a specific cause of death. It suffers from not having a population base in the denominator. Although it does not provide a reliable population estimate as does the cause-specific death rate, it is valuable in making preliminary assessments when denominator data are not available. ■

The next five measures are concerned with events involved in pregnancy, birth, and infancy. Most are based on the number of live births.

*Maternal Mortality Ratio

The **maternal mortality ratio** is defined as the number of deaths assigned to puerperal causes (i.e., those related to childbearing) in a calendar year, divided by the number of live births in that year, the quotient being multiplied by 100,000.

■ **EXAMPLE 7**

United States, 1987—deaths assigned to puerperal causes: 253; live births: 3,829,000.

$$\text{Maternal mortality ratio} = \frac{253}{3,829,000} \times 100,000$$

$$= 6.6 \text{ maternal deaths per } 100,000 \text{ live births per year}$$

Note that this ratio has an inherent problem: It includes maternal deaths in the numerator but only live births in the denominator. Fetal deaths are not represented. Consequently, this practice has a tendency to inflate the ratio slightly. A second problem derives from multiple births. They inflate the denominator but do not affect the numerator. As such events are comparatively rare, the net effect would be a minor change to a ratio based on an otherwise large population. ∎

Infant Mortality Rate

The **infant mortality rate** is defined as the number of deaths of persons of age zero to one in a calendar year, divided by the number of live births in that year, the quotient being multiplied by 1000.

∎ EXAMPLE 8

California, 1987—live births: 494,053; infant deaths: 4546.

$$\text{Infant mortality rate} = \frac{4546}{494,053} \times 1000$$

$$= 9.2 \text{ infant deaths per } 1000 \text{ live births per year} \quad \blacksquare$$

This rate has an inherent problem in those populations that are experiencing rapidly changing birthrates. As you can see from our example, the numerator includes some infants who died in 1987 but were born in 1986; and some of the infants born in 1987 would die in 1988. In a population with a stable birthrate (e.g., that of the United States or Western Europe), such differences are likely to cancel out; this is not the case in a population undergoing a sharp change in its birthrate.

Neonatal Mortality Proportion

The **neonatal mortality proportion** is defined as the number of deaths of neonates (infants less than 28 days of age) that occurred in a calendar year, divided by the number of live births in that year, the quotient being multiplied by 1000.

∎ EXAMPLE 9

California, 1987—deaths at age less than 1 year: 4546; deaths at age less than 25 days: 2780; live births: 494,053.

$$\text{Neonatal mortality proportion} = \frac{2780}{494{,}053} \times 1000$$

$$= 5.6 \text{ neonatal deaths per 1000 live births}$$

Because this example shows that 61.1% [(2780/4546) × 100] of all infant deaths were neonatal, it underscores the importance of neonatal mortality: the great bulk of infant deaths occur in a relatively short period following birth. ■

Fetal Death Ratio

A fetal death is defined as the delivery of a fetus that shows no evidence of life (no heart action, breathing, or movement of voluntary muscles) if the 20th week of gestation has been completed or if the period of gestation was unstated.

The **fetal death ratio** is defined as the number of fetal deaths in a calendar year, divided by the number of live births in that year, the quotient being multiplied by 1000. Note that this ratio applies only to fetal deaths that occur in the second half of pregnancy. No reporting is required for early miscarriages.

■ EXAMPLE 10

California, 1987—fetal deaths: 3477; live births: 494,053.

$$\text{Fetal death ratio} = \frac{3477}{494{,}053} \times 1000$$

$$= 7.0 \text{ fetal deaths per 1000 live births} \quad ■$$

Regrettably, fetal deaths tend to be grossly underreported, so every fetal death ratio is an underestimate (McMillen, 1979).

Perinatal Mortality Proportion

The **perinatal mortality proportion** is defined as the number of fetal plus neonatal deaths, divided by the number of live births plus fetal deaths, the quotient being multiplied by 1000.

■ EXAMPLE 11

California, 1987—fetal deaths: 3477; neonatal deaths: 2780; live births: 494,053.

$$\text{Perinatal mortality proportion} = \frac{3477 + 2780}{3477 + 494{,}053} \times 1000$$

$$= 12.6 \text{ perinatal deaths per 1000 fetal deaths}$$
$$\text{plus live births} \quad ■$$

15.5   MEASURES OF FERTILITY

Measures of fertility are indispensable when approaching population control problems. They are particularly useful in planning maternal and child health services. These measures also help school boards plan their future needs for facilities and teachers. The two most common measures of fertility are the *crude birthrate* and the *general fertility rate*

Crude Birthrate

The **crude birthrate** is defined as the number of live births in a calendar year, divided by the population on July 1 of that year, the quotient being multiplied by 1000.

■ EXAMPLE 12

California, 1987—live births: 494,053; population: 27,663,000.

$$\text{Crude birthrate} = \frac{494,053}{27,663,000} \times 1000$$

$$= 17.9 \text{ live births per 1000 population per year} \quad ■$$

The crude birthrate, although quite commonly used, is a none-too-sensitive measure of fertility because its denominator includes both men and women. Strictly speaking, this measure cannot be a rate because only a fraction of the population is capable of bearing children. A more sensitive measure is the general fertility rate.

General Fertility Rate

The **general fertility rate** is defined as the number of live births in a calendar year, divided by the number of women ages 15–44 at midyear, the quotient being multiplied by 1000.

■ EXAMPLE 13

United States, 1987—live births: 3,829,000; number of women ages 15–44: 58,012,000.

$$\text{General fertility rate} = \frac{3,829,000}{58,012,000} \times 1000$$

$$= 66.0 \text{ live births per 1000 women ages 15–44 per year} \quad ■$$

This rate is more sensitive than the crude birthrate because its denominator includes only women of child-bearing age.

Other measures of fertility are *age-specific fertility rates* and *age-adjusted fertility rates*. Both can be used to make valid comparisons between different population groups.

i.6   MEASURES OF MORBIDITY

At best, mortality data provide indirect means of assessing the health of a community. The **underlying cause of death** hardly provides an adequate picture of the countless illnesses and other health problems that exist in any community. Because morbidity is less precisely recorded than mortality, such data are difficult to analyze, but they are nonetheless useful in program planning and evaluation. Many measures exist. We will discuss here three that deal with the frequency, prevalence, and seriousness of disease.

Incidence Rate

The **incidence rate** is defined as the number of newly reported cases of a given disease in a calendar year, divided by the population on July 1 of that year, the quotient being multiplied by a convenient factor, usually 1000, 100,000, or 1,000,000.

■ **EXAMPLE 14**

California, 1987—new cases of AIDS reported to the State Health Department: 4878; population: 27,663,000.

$$\text{Incidence rate} = \frac{4878}{27,663,000} \times 100,000$$

$$= 17.6 \text{ new cases of AIDS per 100,000 population per year} ■$$

*Prevalence Proportion

The **prevalence proportion** is defined as the number of existing cases of a given disease at a given time, divided by the population at that time, the quotient being multiplied by 1000, 100,000, or 1,000,000.

■ **EXAMPLE 15**

United States, 1988—number of men alive with AIDS: 27,598; population: 120,203,000 men.

$$\text{Prevalence proportion} = \frac{27,598}{120,203,000} \times 100,000$$

$$= 23.0 \text{ AIDS cases per 100,000 men} ■$$

*Case-Fatality Proportion

The **case-fatality proportion** is defined as the number of deaths assigned to a given cause in a certain period, divided by the number of cases of the disease reported during the same period, the quotient being multiplied by 100.

■ **EXAMPLE 16**

United States, 1988—reported number of male AIDS cases: 27,598; deaths from the disease: 13,886.

$$\text{Case-fatality proportion} = \frac{13,886}{27,598} \times 100$$

$$= 50.3\% \text{ mortality among reported cases of AIDS } ■$$

This proportion uses the relative number of deaths as an indicator of the seriousness of a disease. It is often used as a means of showing the relative effectiveness of various methods of treatment.

15.7   ADJUSTMENT OF RATES

Crude rates can be used to make approximate comparisons between different populations. But the comparisons are invalid if the populations are dissimilar with respect to an important characteristic such as age, sex, or race. As we know so well, many diseases have quite different impacts on different groups: on men and women, on old and young persons, on blacks and whites. We would therefore hesitate to compare the death rate for Alaska, with its young population, to that of Florida, with its relatively old population. We can see in Table 15.3 that the crude death rate for Alaska is much lower than that for Florida. The real explanation for this is that Alaska has many more young people than does Florida, and the death rate for a younger group is low. A good way to handle the comparison is to examine the corresponding age-specific death rates for the two states. In this example, Alaska had higher death rates than Florida for five of the six age groups. However, comparing a long series of age-specific rates is often quite cumbersome, especially if more than two populations are involved. To solve this an *adjusted,* or *standardized,* rate is used to make the comparison valid. Statistically, the adjustment removes the difference in composition with respect to age.

There are two methods of adjustment: direct and indirect. The type of data available dictates the method to be used. But keep in mind that an adjusted rate is artificial in that it is a rate applied to a population with a hypothetical distribution. Such rates do not at all reflect the actual rates of a population. They have

Table 15.3 Population Distribution and Age-Specific Death Rates for Alaska and Florida, 1987

| | Alaska | | | | Florida | | | |
| | | Population | | | | Population | | |
Age Group	Number of Deaths	Persons	%	Deaths per 100,000 Persons	Number of Deaths	Persons	%	Deaths per 100,000 Persons
0–4	163	60,000	11.45	271.7	2271	812,000	6.75	279.7
5–24	152	173,000	33.01	87.9	2296	3,093,000	25.73	74.2
25–44	376	193,000	36.83	194.8	6958	3,450,000	28.70	201.7
45–64	518	79,000	15.08	655.7	20,524	2,528,000	21.03	811.9
65+	845	19,000	3.63	4447.4	95,141	2,139,000	17.79	4447.9
Total	2054	524,000	100.00	392.0	127,290	12,022,000	100.00	1,058.8

SOURCE: *1990 Statistical Abstracts of the United States.*

real meaning only as relative comparisons. The numerical values of the adjusted rates depend in large part on the choice of the standard population.

The Direct Method

The **direct method of adjustment** applies a standard population distribution to the death rates of two comparison groups. The sum of the expected deaths for the two groups is then used to compute the adjusted death rate (dividing the expected deaths by the total of the standard population). For the direct method it is essential to have both the age-specific death rates for the populations being adjusted and the distribution of the standard population by age (or by whatever other factor is being adjusted).

■ EXAMPLE 17

In Table 15.3, we see that the 1987 crude death rate per 100,000 population for Alaska was 392.0 and for Florida, 1058.8. But a close look at the age distribution discloses that Alaska had a higher percentage of its population in the younger age groups. This finding makes it essential to adjust the death rates of the two states in order to make a valid comparison. With the direct method, we can figure out what the death rate would be for each state if the age distributions of both populations were identical. An efficient way to make this calculation is to apply the U.S. standard population to both states and then compute the expected number of deaths for each state as if its population distribution were indeed the same as for the U.S. standard.

To carry out this method, we use the **U.S. standard million.** This is a population of 1 million persons that identically follows the age distribution for the entire United States, as shown in column 1 of Table 15.4.

Table 15.4 Age-Adjusted Death Rates per 100,000 Population for Alaska and Florida (1987) Using the Direct Method and Based on the 1987 U.S. Standard Million

Age Group	(1) 1987 U.S. Standard Million	(2) Alaska Age-Specific Death Rates	(3) Alaska Expected Deaths with U.S. Standard Million	(4) Florida Age-Specific Death Rates	(5) Florida Expected Deaths with U.S. Standard Million
0–4	75,080	271.7	204.0	279.7	210.0
5–19	216,113	87.9	190.0	74.2	160.4
20–44	400,170	194.8	779.5	201.7	807.1
45–64	186,091	655.7	1220.2	811.9	1510.9
65+	122,546	4447.4	5450.1	4447.9	5450.7
Total	1,000,000		7843.8		8139.1

The specific steps involved in calculating the age-adjusted rate are as follows:

1. Compute the expected number of deaths for the standard population by applying the age-specific death rates of the state. For Alaska, multiply column 1 by column 2, divide the product by 100,000, and enter the result in column 3. For Florida, multiply column 1 by column 4, divide the product by 100,000, and enter the result in column 5.

2. Total the expected deaths in columns 3 and 5. You can see that if Alaska's population were distributed the same as the standard million, the expected number of deaths (given Alaska's known age-specific death rates) would be 7843.8. Similarly, the expected number of deaths for Florida would be 8139.1.

3. Compute the age-adjusted death rate per 1000 by dividing the total expected deaths by 1000. For Alaska, the adjusted rate is 7.84, and for Florida it is 8.14. Remember the crude death rates (Table 15.3) were 3.92 for Alaska and 10.59 for Florida.

The striking result: Florida's crude death rate was much higher than Alaska's. However, where based on a comparable population, the age-adjusted death rates were nearly the same for both states! ■

The choice of the standard population affects the values of the adjusted rates. Therefore, in comparing adjusted rates between different states or countries, you should know which standard population was used because different standards will yield different results.

The Indirect Method

The **indirect method of adjustment** is somewhat different from the direct method. It is utilized when age-specific death rates are not available for the

populations being adjusted, but when the age-specific death rates for the standard population are known. With this method, we compute a **standard mortality ratio** (SMR) (observed deaths divided by expected deaths) and use it as a standardizing factor to adjust the crude death rates of the given populations. The SMR increases or decreases a crude rate in relation to the excess or deficit of the group's composition as compared to the standard population. A detailed treatment appears in several textbooks. See, for instance, Remington and Schork (1985) or Lilienfeld, Pedersen, and Dowd (1967). Both this method and the direct method are as applicable to ratios and proportions as to rates.

Conclusion

Public health decision making is a quantitative matter. The health of a population is assessed by use of its vital statistics and demographic data. Information about demographic characteristics is obtainable from census data, registration of vital events, and morbidity surveys. Such data are used to calculate vital rates and other statistics that are used to indicate the magnitude of health problems.

Vital rates, ratios, and proportions are classified into measures of mortality (death), fertility (birth), and morbidity (illness). These measures may be crude or specific, the latter referring to calculations for subgroups selected for a common characteristic such as age, sex, race, or disease experience. Comparisons of vital rates, ratios, or proportions among different populations should be made with care and be validated by use of specific or adjusted measures. Choice of the adjustment method depends on the type of data available.

Vocabulary List

adjusted rate
  (standardized rate)
base
birth registration
case-fatality proportion
crude birthrate
crude rate
death registration
decennial census
demographic variables
direct method of
  adjustment
fetal death ratio
general fertility rate
incidence rate

indirect method of
  adjustment
infant mortality rate
maternal mortality ratio
Metropolitan Statistical
  Area (MSA)
morbidity data
mortality data
National Health Survey
neonatal mortality
  proportion
perinatal mortality
  proportion
population at risk
prevalence proportion

proportion
proportional mortality
  ratio
rate
ratio
specific rate
standard mortality ratio
  (SMR)
time at risk
underlying cause of
  death
U.S. standard million
vital events
vital statistics

Exercises

Note: For all these exercises, use as appropriate the sources referred to in Section 15.2.

15.1   Find the size of the U.S. population (including those in the armed forces) for 1970, 1980, and 1990.

15.2   What was the population of New York State in 1970? In 1980?

15.3   In 1987, how many Iowans were
a. under 5 years old
b. 65 or more years old

15.4   What was the percentage of blacks living in 1987 in Minnesota? In Georgia?

15.5   In 1987, what were the birth and death rates for Alaska? For Kansas?

15.6   For the United States during 1987, what were the five leading causes of death?

15.7   What were the maternal mortality ratios for U.S. whites and nonwhites in 1950? In 1980?

15.8   What were the death rates from cirrhosis of the liver by sex and race (white and nonwhite) for the United States in 1987?

15.9   What were the numbers of total deaths, infant deaths, and neonatal deaths, by place of residence, for two California counties, Riverside and San Bernardino, in 1987?

15.10 a. For 1987, compute the crude birthrates for Alaska and for Arizona.
b. What do you observe about the birthrates of these two states? What are some possible explanations?

15.11 Find the state death rate for Hawaii and Nevada. Which state had the highest birth and fertility rates in 1990?

15.12 Obtain the sex-specific death rates from cirrhosis of the liver for males and females in 1993 for the United States.

15.13 Obtain the following cause-specific death rates for Michigan, Utah, Tennessee, and the United States for cancer, heart disease, accidents, and diabetes in 1993.

15.14 Obtain the population size and number of deaths due to cancer in Tennessee for 1980 and 1990.

15.15 Find the states with the three highest HIV death rates in 1993.

16 Life Tables

Learning Objectives

After studying this chapter, you should be able to

1. Distinguish among the three types of life tables

2. Identify and be able to compute the components of a current life table

3. Compute measures of mortality and longevity from a life table

4. Construct a follow-up life table

5.1 INTRODUCTION

Life tables have been in use for centuries. The first systematic, if inexact, life table was developed by British astronomer Edmund Halley (of Halley's comet fame) to describe the longevity of residents of seventeenth-century Breslau. In 1815, Joshua Milne published the first mathematically accurate life table, which described the mortality experience of a city in northern England (Shyrock and Siegel, 1973).

Life tables are now in general use and have many important applications. For instance, they are used by demographers to measure and analyze the mortality or longevity of a population or one of its segments; by insurance companies to compute premiums; and by research workers to determine whether the differences in mortality or longevity of two groups are different. They are employed to predict survival or the likelihood of death at any time. A life table analysis

can be fundamental to the solution of many public health and medical problems.

Three types of life tables are in general use. They are the *current* life table, the *cohort*, or *generation*, life table, and the *follow-up*, or *modified*, life table. Current and follow-up life tables are the most common and will be discussed in some detail.

The **current life table** illustrates how age-specific death rates affect a population. Such a table considers mortality rates for the entire population for a given period. For instance, a 1979–1981 life table considers the mortality of the various age groups over three years. It does not follow the mortality experience of a single age group throughout its life. Three years are used in preference to one year because this span tends to stabilize the death rates, which otherwise would be unduly sensitive to year-by-year fluctuations.

By contrast, the **cohort life table** follows a defined group (**cohort**) from birth (or some other measurable point in time) until the last person in the group has died, which is why it is also known as a **generation life table.** The key difference between the current life table and the cohort life table is that the former generates a fictitious pattern of mortality, whereas the latter presents the historical record of what actually occurred.

Because there are usually major differences in the patterns of mortality among various subgroups of a population, life tables are quite commonly constructed for specific groups: by race, sex, occupation, or specific diseases.

An interesting extension of the life-table idea has come into general use in recent years. Life tables may be employed for studies wherein the outcome variable is an event other than death. For example, an outcome could be recurrence of coronary heart disease, a contraceptive failure, or time from driver's license application to first reported accident. An illustration of the recurrence of cancer as an outcome variable appears in Kuzma and Dixon (1986). Furthermore, the **follow-up,** or **modified, life table** has been used recently by medical researchers. They have adopted its use to determine the survival experience of patients with a particular condition.

16.2   CURRENT LIFE TABLES

To demonstrate the many applications of a current life table, we will use an **abridged life table** for the 1987 U.S. population. Table 16.1 illustrates what would have happened to a hypothetical population of 100,000 persons as it passed through time—that is, how many persons would have died and how many would have survived in each particular age group if they spent their entire lifetimes exposed to the 1987 mortality rate. The table also indicates the probability of dying during any age interval, the probability of surviving to a particular age, and the average life expectancy. The table is *abridged* for

Table 16.1 Abridged Life Table for the Total U.S Population, 1987

Age Interval	Proportion Dying			Of 100,000 Born Alive		Person-Years Lived		Average Remaining Lifetime
Period of Life Between Two Exact Ages Stated x to $(x + n)$	Uncorrected — Average Annual Age-Specific Death Rate $_nm_x$	Correction Term — Fraction of Last Age Interval Lived $_na_x$	Corrected — Proportion Dying During Age Interval $_n\hat{q}_x$	Number Living at Beginning of Age Interval l_x	Number Dying During Age Interval $_nd_x$	In the Age Interval $_nL_x$	In This and All Subsequent Age Intervals T_x	Average Number of Years of Life Remaining at Beginning of Age Interval $\hat{e}_x$
<1	.0101724	.10	.0100801	100,000	1,008	99,093	7,510,914	75.11
1–4	.0005149	.39	.0020570	98,992	204	395,472	7,411,821	74.87
5–9	.0002404	.46	.0012012	98,788	119	493,620	7,016,349	71.02
10–14	.0002877	.56	.0014375	98,669	142	493,035	6,522,729	66.10
15–19	.0008659	.57	.0043214	98,527	426	491,720	6,029,694	61.20
20–24	.0011492	.49	.0057292	98,101	562	489,070	5,537,974	56.45
25–29	.0012114	.50	.0060753	97,539	593	486,215	5,048,904	51.76
30–34	.0014645	.52	.0072968	96,946	707	483,035	4,562,689	47.06
35–39	.0018554	.54	.0092375	96,239	889	479,150	4,079,654	42.39
40–44	.0025259	.54	.0127201	95,350	1,213	473,960	3,600,504	37.76
45–49	.0038527	.54	.0190943	94,137	1,797	466,550	3,126,544	33.21
50–54	.0062165	.53	.0306349	92,340	2,829	455,050	2,659,994	28.81
55–59	.0097092	.52	.0474405	89,511	4,246	437,365	2,204,944	24.63
60–64	.0152116	.52	.0733790	85,265	6,257	411,310	1,767,579	20.73
65–69	.0223704	.52	.1061527	79,008	8,387	374,910	1,356,269	17.17
70–74	.0342433	.51	.1579639	70,621	11,156	325,775	981,359	13.90
75–79	.0511817	.51	.2273943	59,465	13,522	264,195	655,584	11.02
80–84	.0795990	.48	.3297506	45,943	15,150	190,325	391,389	8.52
85+	.1531499	—	1.0000000	30,793	30,793	201,064	201,064	6.53

SOURCE: *Monthly Vital Statistics Report* (National Center for Health Statistics) 1986–1988.

convenience, most of the age intervals covering five-year periods. A **complete life table** would have a separate entry for each year.

By systematically dissecting a life table, we can gain some valuable insights into what it means and how it works. We will begin by discussing the several columns of Table 16.1.

Age Interval [x to $(x + n)$]

The **age interval** is the period between the two exact ages stated. For example, 35–40 means the five-year span between the 35th birthday and the 40th.

Age-Specific Death Rate ($_n m_x$)

The symbol $_n m_x$ denotes the average annual age-specific death rate for the age interval stated; that is, x denotes the beginning of the interval and n denotes the width. The numerator for this rate is the average number of deaths per year in a three-year period (1986–1988), divided by the July 1 average population for 1986, 1987, and 1988. For example, the age-specific death rate for age group 35–40, $_5 m_{35}$, is .0018554, or about 1.9 per 1000.

Correction Term ($_n a_x$)

We need a correction term for a very simple reason. Among tiny infants, most deaths occur early in the first year, whereas among adults deaths are fairly uniformly distributed throughout the year. The correction term defines and adjusts for the maldistribution. The $_n a_x$ column shows the average fraction of the age interval lived by persons who die during that interval. Notice that $_5 a_{35}$ is .54, a shade more than half a year. Values for $_n a_x$ are computed by use of a complex equation discussed in advanced treatments of this topic such as in Chiang (1984).

Corrected (Estimated) Death Rate ($_n \hat{q}_x$)

The symbol $_n \hat{q}_x$ denotes the proportion of those persons who are alive at the beginning of the age interval but die during that interval. For example, the probability that a 35-year-old will die before reaching 40 is $_5 \hat{q}_{35} = .0092$. The computing formula for $_n \hat{q}_x$ is

$$_n \hat{q}_x = \frac{n \cdot {_n m_x}}{1 + (1 - {_n a_x}) \cdot n \cdot {_n m_x}}$$

Number Living at Beginning of Age Interval (l_x)

We use l_x to indicate the number of persons, starting with the original cohort of 100,000 live births, who survive to the exact age marking the beginning of each interval. Each l_x value is computed by subtracting the $_n d_x$ (number dying during interval) for the previous age interval from the l_x for that interval—that is,

$$l_{x+n} = l_x - d_x$$ (16.1)

Thus

$$l_{35} = l_{30} - {}_5d_{30} = 96{,}946 - 707 = 96{,}239$$

Number Dying During Age Interval ($_nd_x$)

The number of persons of the original 100,000 who die during each successive age interval is denoted by $_nd_x$. It is calculated by applying the proportion dying ($_n\hat{q}_x$) during the interval to the number alive (l_x) at the beginning of the interval.

$$_nd_x = (l_x)(_n\hat{q}_x)$$ (16.2)

For example,

$$_5d_{30} = (l_{30})(_5\hat{q}_{30}) = 96{,}946(.0072968) = 707$$

Person-Years Lived in Interval ($_nL_x$)

The symbol $_nL_x$ designates the totality of years lived by the survivors of the original 100,000 (the l_x) between the ages x and $(x + n)$. For example, $_5L_{30} = 483{,}035$ is the number of **person-years** lived by the 96,946 (l_{30}) alive at the beginning of the 30th year. It is computed by the equation

$$_nL_x = n[l_{x+n} + (_na_x)(_nd_x)]$$ (16.3)

for all intervals except the last, for which

$$_nL_x = \frac{_nd_x}{_nm_x}$$ (16.4)

So

$$\begin{aligned}
_5L_{30} &= 5[l_{35} + (_5a_{30})(_5d_{30})] \\
&= 5[96{,}239 + (.52)(707)] \\
&= 483{,}033
\end{aligned}$$

Sometimes the $_nL_x$ column of the life table is termed the **stationary population.** Given the hypothetical assumption that the number of births and deaths remains constant each year, the number of person-years would in fact be unchanging, hence the term. This idea is useful in certain applications to studies of population structure.

Total Number of Person-Years (T_x)

The symbol T_x denotes the total number of person-years lived by the l_x survivors from year x to death. It is obtained by cumulating the person-years lived in the intervals ($_nL_x$):

$$T_0 = {}_1L_0 + {}_4L_1 + {}_5L_5 + \cdots + {}_5L_{80} + {}_5L_{85} = 7,510,914$$

Expectation of Life ($\hat{e}_x$)

Because of its general usefulness, $\hat{e}_x$ may be the most valuable feature of the life table. It denotes **life expectation,** the average number of years of life remaining to those who survive to the beginning of the age interval. It is calculated by dividing the number of person-years lived after a given age (T_x) by the number who reached that same age (l_x):

$$\hat{e}_x = \frac{T_x}{l_x} \tag{16.5}$$

The future life expectancy for a 35-year-old, for example, is calculated by $\hat{e}_{35} = T_{35}/l_{35} = 4,079,654/96,239 = 42.4$ years; that is, on average, persons reaching age 35 may expect to live to $35 + 42.4 = 77.4$ years.

A life table enables us to compute some special measures of mortality that are real improvements over the use of general rates. One of these measures is the *expectation of life at age 1,* which removes the considerable impact that infant mortality has on life expectation from birth. Another is the *expectation of life at age 65,* which zeros in on the mortality of the older ages when most deaths occur. Still another is the *probability of surviving from birth to age 65,* which is defined as

$$_{65}P_0 = \frac{l_{65}}{l_0}$$

An interesting measure is the **median age at death,** which is the age to which precisely half of the cohort survives. It corresponds to the age x at which $l_x = 50,000$ in a life table based on a cohort of 100,000 persons. By interpolation from Table 16.1, we would estimate the median age at death as 78.5 years.

A commonly used survival rate in population studies is

$$_np_x = \frac{l_{x+n}}{l_x} \tag{16.6}$$

the probability of surviving from year x to year $x + n$. For example, using Table 16.1, we can calculate the proportion of newborn babies who will reach their tenth birthday:

$$_{10}p_0 = \frac{l_{10}}{l_0} = \frac{98,669}{100,000} = .98669$$

Similarly, the proportion of newborns who will reach their first birthday is

$$_{1}p_0 = \frac{l_1}{l_0} = \frac{98,992}{100,000} = .98992$$

and the probability that a 25-year old will survive 10 more years is

$$_{10}p_{25} = \frac{l_{35}}{l_{25}} = \frac{96,239}{97,539} = .98667$$

Not surprisingly, we can follow the same pattern to compute probabilities of death. The probability that a 25-year-old will die before reaching age 30 is

$$_{5}q_{25} = \frac{_{5}d_{25}}{l_{25}} = \frac{593}{97,539} = .00608$$

Note that l_x may be thought of as a cumulation of the *age-specific death rates up to (but not including) age x*. In other words, it shows the net effect of all death rates up to that age, whereas life expectation, $\hat{e}_x$, shows the effect of the age-specific death rates after that age.

We already mentioned that the current life table considers a hypothetical cohort. The assumption is that the cohort is subject throughout its existence to those age-specific mortality rates that were observed for one particular period. However, specific rates actually vary with time. Although little variation occurs from one year to the next, significant changes are common over long periods. Table 16.2 illustrates the point for U.S. white males for the years 1900–1980. Note that most of the improvement in longevity has occurred under age 65, and especially in the first year of life.

Table 16.2 Changes in the Mortality of White Males in the United States According to Various Life Table Measures, 1900–1980

Measure	Base Period for Life Table*						
	1900	1910	1920	1930	1940	1950	1980
Expectation of life at birth	48.2	50.2	56.3	59.1	62.8	66.3	73.6
Expectation of life at age 1	54.6	56.3	60.2	62.0	65.0	67.1	73.6
Expectation of life at age 65	11.5	11.3	12.2	11.8	12.1	12.8	16.4
Probability of surviving from birth to age 65	.39	.41	.51	.53	.58	.64	.77
Median age at death of initial cohort	57.2	59.3	65.4	66.4	68.7	70.7	77.1

*Life tables for periods before 1929–1931 relate to those states that required death registration.

Demographers make an important distinction between **life span** and **life expectation.** A life span of "four score years and ten" has been well known from time immemorial. Although inexact, life span is the age that persons are likely to reach, given optimum conditions. Life span could be defined as that age reached by the longest-lived 0.1% of the population, which would currently be quite close to 100 years (Shyrock and Siegel, 1973). Life expectation has increased not so much by virtue of a longer life span as by reduction of infant mortality, and thus an increase in the average years of life.

16.3   FOLLOW-UP LIFE TABLES

"How long do I have?" is often the first question a patient asks the physician when told that he or she is suffering from a life-threatening chronic disease. The **follow-up life table** (or **modified life table**) provides a basis for answering this difficult question. Chronic-disease registries, especially cancer registries, make regular use of the follow-up table to track the survival of patients over time. In this connection, life tables are often used to evaluate the relative effectiveness of alternative modes of treatment by computing the probability of survival of patients treated by each mode.

The follow-up table is particularly useful because it utilizes the experience of each person for the entire time he or she was in the study; that is, the method considers the period of exposure in terms of person-years or other appropriate units.

Life tables may be calculated for a cohort in which all the members start the study at the same time or for one in which the members are admitted to the study at different times over a period of years. In either case, the data are handled identically, providing that (1) death rates do not change materially over time, and (2) exposure to the disease prior to treatment is not increasing with time.

Construction of a Follow-up Life Table

To construct a follow-up life table, you will need to know the period of follow-up after some event, such as a heart attack, diagnosis of cancer, or surgery. To ensure accuracy, you need well-defined starting and end points. Given a known period of observation for each patient, you can then tally how many survive, how many die, and how many are lost to follow-up during the first and subsequent years of the study.

The construction of such a table is illustrated in Table 16.3 with data from a cancer follow-up study. A total of 356 (l_0) patients began the study. During the first year of follow-up, 60 (d_0) patients died. Thus, the probability of surviving the first year was $\hat{p}_1 = (356 - 60)/356 = .8315$. During the second year, of the 296 patients remaining, 47 died; 1 was lost to follow-up. By convention, it is as-

Table 16.3 Follow-up Life Table: Classification of Cases and Survival Rates of Cancer Patients

Interval in Years x to $(x + 1)$	Alive at Beginning of Interval l_x	Died During Interval d_x	Lost to Follow-up f_x	Withdrawn Alive w_x	Effective No. Exposed to Risk of Dying l'_x	Proportion Dying $\hat{q}_x$	Proportion Surviving $\hat{p}_x$	Survival Rate P_{0x}
0–1	356	60	0	0	356	0.1685	.8315	.8315
1–2	296	47	1	0	295.5	0.1591	.8408	.6992
2–3	248	29	5	0	245.5	0.1181	.8818	.6166
3–4	214	24	20	25	191.5	0.1253	.8746	.5393
4–5	145	11	13	50	113.5	0.0969	.9032	.4871
5–6	71	4	0	57	42.5	0.0941	.9057	.4412

sumed that a person who is lost to follow-up (f_x) or who withdraws from the study alive (w_x) lives through half the interval. Consequently, the effective number exposed to the risk of dying is here estimated as $l'_x = 296 - .5 = 295.5$. The probability of surviving the second year of follow-up is then estimated as

$$\hat{p}_2 = \frac{l'_2 - d_2}{l'_2} = \frac{295.5 - 47}{295.5} = .8409 \tag{16.7}$$

The probabilities of surviving successive years are computed similarly. The equation for the effective number exposed to the risk of dying may be summarized as

$$l'_x = l_x - .5(w_x + f_x) \tag{16.8}$$

Having found the probabilities of survival for each individual year, we can now easily compute the probability of surviving several years. For example, the probability of surviving the first two years is $P_{02} = (p_1)(p_2)$, and the first five years is $P_{05} = (p_1)(p_2)(p_3)(p_4)(p_5)$.

The **five-year survival rate** is commonly used in cancer research as a measure of a treatment's effectiveness. Differences between survival rates of two groups are tested by means of a t test, which implies the need to know standard errors. For a detailed discussion of two different methods of preparing a life table, see Kuzma (1967).

Some special problems in calculating survival rates occur when persons are lost to follow-up or withdraw alive (i.e., persons are known to be alive at the beginning of the time interval, but their fate is unknown at the end). Numerous suggestions have been offered on how to handle these problems. For instance, if the proportion of such cases is small, the assumption is made that each case was lost or withdrew at the middle of the last known interval. Thus, the convention is that such cases are considered to be alive for half of the last interval during which they were observed.

Clinical trials frequently utilize life tables to estimate survival rates. It is often necessary to determine whether there is a statistically significant difference between P'_{0x}, the xth year survival rate of a treatment group, and P_{0x}, the xth year survival rate of a control group. The equation used is

$$Z = \frac{P'_{0x} - P_{0x}}{\sqrt{SE(P'_{0x})^2 + SE(P_{0x})^2}} \tag{16.9}$$

where $SE(P'_{0x})$ and $SE(P_{0x})$ are standard errors for the two groups and Z is the normal deviate.

A rigorous justification for the standard-error equation is beyond the scope of this book. However, an approximation suggested by M. Greenwood as described in Cutler and Ederer (1958) is as follows:

$$SE(P_{0x}) = P_{0x} \sqrt{\sum \frac{d_x}{(l_x - \frac{1}{2} w_x)(l_x - d_x - \frac{1}{2} w_x)}} \qquad (16.10)$$

where x is summed from $x = 0$ to $x = n$; that is, through the interval prior to that containing P_{0x}.

We illustrate the computation of the standard error of P_{05} using the data from Table 16.3:

$$SE(P_{05}) = .4412 \sqrt{\frac{60}{(356)(296)} + \frac{47}{(296)(249)} + \frac{29}{(248)(215)} + \frac{24}{(214)(189.5)} + \frac{11}{(145)(109)}}$$

$$= .4412 \sqrt{.0005693 + .0006376 + .0005438 + .0005918 + .0006959}$$
$$\qquad (16.11)$$
$$= .4412 \sqrt{.0030384} = .4412(.055122)$$

$$= .02432$$

Using the value of the standard error, it is now possible to compute the z statistic.

Conclusion

Life tables provide excellent means for measuring mortality and longevity. The current life table shows the effects of age-specific death rates on a group. From this table, measures of mortality and life expectation can be computed. Whereas the current life table presents a hypothetical picture of the effects of present mortality rates, the cohort life table is an actual historical record of the mortality of a group followed through life. The follow-up life table considers the experience of persons from event to event during the period of a study.

Vocabulary List

abridged life table	current life table	life table
age interval	five-year survival rate	median age at death
cohort	follow-up life table	person-years
cohort life table	(modified life table)	stationary population
(generation life table)	life expectation	
complete life table	life span	

Exercises

16.1   Table 16.4 is an incomplete abridged life table for the U.S. population (1980). Complete the table by filling in the blanks.

16.2   A distinguished citizen is celebrating his 75th birthday. Use Table 16.1 to compute the probability that he will live to celebrate his 80th.

In Exercises 16.3 through 16.9 use your completed life table from Exercise 16.1.

Table 16.4 Abridged Life Table for the Total U.S. Population, 1980

	Proportion Dying			Of 100,000 Born Alive		Person-Years Lived		Average Remaining Lifetime
Age Interval	Uncorrected	Correction Term	Corrected	Number Living at Beginning of Age Interval	Number Dying During Age Interval	In the Age Interval	In This and All Subsequent Age Intervals	Average Number of Years of Life Remaining at Beginning of Age Interval
Period of Life Between Two Exact Ages Stated x to $(x+n)$	Average Annual Age-Specific Death Rate $_nm_x$	Fraction of Last Age Interval Lived $_na_x$	Proportion Dying During Age Interval $_n\hat{q}_x$	l_x	$_nd_x$	$_nL_x$	T_x	$\mathring{e}_x$
<1	.0127445	.10	.0126000	100,000	1,260	98,866	7,361,560	73.62
2–4	.0006510	.39	.0025999	98,740	257	394,334	7,262,690	73.55
5–9	.0003403	.46	.0016999	98,483	167	491,964	6,868,360	69.74
10–14	.0003002	.56	.0015000	98,316	147	491,255	6,376,390	64.86
15–19	.0010222	.57	.0050998	98,168	501	489,766	5,885,140	59.95
20–24	.0013647	.49	.0067998	97,668	664	486,645	5,395,370	55.24
25–29	.0013646	.50	.0067998	97,004	660	483,369	4,908,730	50.60
30–34	.0014651	.52	.0072998	96,344	703	480,032	4,425,360	45.93
35–39	.0017873	.54	.0088999	95,641	851	476,246	3,945,330	41.25
40–44	.0028181	.54	.0139998	94,790	1,327	470,896	3,469,080	36.60
45–49	.0045062	.54	.0222999	93,463	2,084	462,519	2,998,180	32.08
50–54	.0072412	.53	.0356002	91,378	3,253	449,247	2,535,660	27.75
55–59	.0111928	.52	.0545000	88,125	4,803	429,099	2,086,420	23.68
60–64	.0169420	.52	.0814002	83,322	6,782	400,334	1,657,320	19.89
65–69	.0246352	.52	.1163000	76,540	8,902	361,336	1,256,980	16.42
70–74	.0363293	.51	.1668000					
75–79	.0608562	.51	.2648000					
80–84	.0894622	.48	.3629000			171,806	171,806	6.51
85+	.1536440	—	1.0000000	26,397	26,397			

16.3   Calculate the probability at birth of living to be 80.

16.4   Compute the following proportions:
     a. all persons dying between birth and the first birthday
     b. all persons dying between birth and the fifth birthday
     c. babies born alive dying between birth and the fifth birthday

16.5   Find the proportion of
     a. all persons dying between the ages of 35 and 45
     b. 35-year-olds dying between the ages of 35 and 45

16.6   What is the probability that a person aged 20 will survive until age 65?

16.7   Find the expectation of life at birth, at 1 year, at 35 years, and at 75 years of age.

16.8   Find the proportion of 70-year-olds dying between the ages of 70 and 75. Compare this figure with that found in Exercise 16.4c and explain the difference.

16.9   Why are the results of (b) and (c) of Exercise 16.4 the same and the results of (a) and (b) of Exercise 16.5 different?

17 The Health Survey and the Research Report

Chapter Outline

17.1 Planning a Health Survey
Presents an outline for a survey with a brief discussion of the steps involved

17.2 Evaluation of a Research Report
Lists and discusses steps for evaluating a medical report

Learning Objectives

After studying this chapter, you should be able to

1. Prepare an outline for a health survey
2. Be prepared to critically evaluate a medical report

17.1   PLANNING A HEALTH SURVEY

So far in this book we have discussed the kinds of statistical topics generally covered by most introductory statistics textbooks. In this section, we consider the survey, one of two research tools that are indispensable to persons who deal with data and statistics. In the next section, we discuss the second tool, the evaluation of research articles. The coverage of both topics is all too brief since each could itself be the subject of a good-sized book.

Health surveys are conducted for a number of reasons, but most often they are undertaken to determine the health needs of a community. Subjects of a health survey are members of the general public, all of whom are, to some degree, users of health services. In the same sense that people consume gasoline, stockings, and corn flakes, they are regarded as **consumers** of health services.

What constitutes a health survey? Many things. For instance, health surveys may entail inquiries into the consumer's knowledge, attitudes, and practices, his or her utilization of health services, disease experience in the past, and satisfaction (or dissatisfaction) with health service delivery; or they may involve research directed ultimately toward elucidating the etiology of a disease or

evaluation of a program's success. All these points are generally focused in one direction: toward aiding the decision-making process of health service (or public health) managers.

The goal of most researchers is to be able to conduct a survey that clearly and accurately describes some health-related phenomenon. But caution is advised; a health survey can be a tricky business. Unless it follows a prescribed stepwise procedure (like the one we outline here), a survey could produce faulty information leading to unfortunate (possibly grave) consequences. An example of a report with an intriguing finding—that left-handed people have a shortened life expectancy—was reported in the *New England Journal of Medicine*. With time, we will learn how well it stands up to careful scientific scrutiny.

Immediately after the following outline, each step will be briefly described.

*Outline for Planning a Health Survey**

1. Make a written statement of the purpose of the survey.
2. Write out the objectives and hypotheses.
3. Specify the target population.
4. List the variables to be measured.
5. Review existing pertinent data.
6. Outline the methods of data collection.
7. Establish the time frame.
8. Design the questionnaire.
9. Pretest the questionnaire.
10. Select subjects for the sample.
11. Collect the data.
12. Edit, code, and enter the data on a computer and verify the data entry.
13. Analyze the data.
14. Report the findings.

Step 1: Make a Written Statement of the Purpose

The purpose of your survey should be well thought out, carefully defined, and clearly stated in two or three sentences. This step will aid your own thinking and will help you in carrying out the subsequent steps. Without it, a survey is doomed to failure.

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*Credit for this outline goes to Dr. David Abbey, a survey statistician who developed it for a course on Health Survey Methods.

Step 2: Formulate Objectives and Hypotheses

A descriptive survey seeks to estimate one or more characteristics of a population. That, quite simply, is its specific objective. An analytical survey seeks to examine relationships among some specified characteristics. To carry it out, you need to define the hypotheses to be tested.

Step 3: Specify the Target Population

The **target population** is that group of people from whom inferences are to be drawn. This population may well be restricted to one from which the investigator may feasibly draw a sample. To test your research hypothesis, it is essential to estimate certain key characteristics of individual members of the target population. In statistical sampling, an individual member of a population is often referred to as an **element.** But in health surveys the element may be a person, a mother–child pair, or some logical group of persons such as a household. Measurements are taken on the element. The population can be defined as the collection of all elements.

Once your target population is defined and elements are identified, list the variables that are to be assessed on each element. For example, a target population might be all the students enrolled in a college course who successfully stopped smoking during the last 12 months. The element would be each member of the class possessing that characteristic; variables measured might be age, sex, amount of smoking, and number of years of smoking before quitting.

Step 4: List the Variables

There is an endless list of potential variables that you may wish to measure. In general, the researcher focuses on personal characteristics of individual members of the target population. Such characteristics could be a person's weight, blood pressure, age, race, smoking status, and so on. The variables considered should be potentially measurable on each person. In a health survey, one usually wants to collect information both on outcome variables and on concomitant variables. The latter are those covariables that, although themselves uncontrollable, may well affect the outcome. All variables should be clearly defined during the planning stages.

Step 5: Review Existing Data

It is important to review current literature on the topic being surveyed so that you can determine the state of the art, current hypotheses, those variables regarded as pertinent, and the likely success of your chosen strategy. It is often advisable to use standardized questions for which ample documentation of validity and reliability exists. And by using the standard wording of standardized questions, you will be able to compare results with those of well-known studies.

Step 6: Decide How to Collect Data

In collecting data, there are numerous methods to choose from, each of which has certain advantages and disadvantages. The **person-to-person interview** is often regarded as the industry standard because of the high response rate. The interviewer, being at the scene, can make additional observations regarding subtle aspects of the interviewee's behavior; these may be used to help validate the interview. But this approach is costly; an alternative is the **telephone interview** (using random-digit dialing), which can be performed at approximately half the cost and produce essentially similar results. But telephoning introduces a new potential bias in that it excludes approximately 10% of the households—those that do not have a telephone or have an unlisted number. Still another approach is the **mailed questionnaire.** This costs less than person-to-person or telephone interviews and may be done anonymously. Mailed questionnaires rule out the problem of **interviewer bias;** the respondent is less likely to be defensive about answering socially sensitive questions. But this approach does have a serious drawback: poor response rates. It usually requires at least two follow-up mailings to obtain a satisfactory number of responses. Other problems include uncertainty as to whether the intended person actually completed the questionnaire, plus nagging doubts as to whether the respondents are truly representative of the target population.

Step 7: Establish the Time Frame

It is necessary to establish a time frame to realistically schedule survey events. The schedule should not be so tight as to jeopardize succeeding steps in case of a delay in preceding events. Plan for backup procedures and personnel to avoid major delays. It is a good idea to have some trained interviewers on call.

Step 8: Design the Questionnaire

Questions need to be carefully worded so as not to confuse the respondent or arouse extraneous attitudes. The questions should provide a clear understanding of the information sought. Be precise; avoid ambiguity and wording that might be perceived to elicit a specific response. Questions may be open-ended, multiple choice, completion, or a variation of these. You should studiously avoid overly complex questions. The key principles to keep in mind while constructing a questionnaire are that it should (1) be easy for the respondent to read, understand, and answer; (2) motivate the respondent to answer; (3) be designed for efficient data processing; (4) have a well-designed professional appearance; and (5) be designed to minimize missing data.

Step 9: Pretest the Questionnaire

It is never possible to anticipate in advance all the potential problems that may occur when you administer a questionnaire. So it is important to **pretest** it. A

pretest will identify questions that respondents tend to misinterpret, omit, or answer inappropriately. It should be done on a handful of individuals similar to, but not included in, the target population and should utilize the same methodology that will be used in the actual survey.

Step 10: Select the Sample

You should select the sample in such a way that valid statistical inferences can be drawn regarding the target population. You wish to obtain a representative sample, one that minimizes **sampling bias** and is designed for economy in operation. A variety of sampling designs are available: simple random, systematic random, stratified random, and multistage sampling. To determine the most appropriate design for a complex survey, consult a survey statistician.

Step 11: Collect the Data

With a completed and pretested questionnaire, you are ready for data collection. This step requires careful planning and supervision to ensure data of good quality. You want to attain the following objectives: maximize the response rate by minimizing nonresponses, keep track of the nonrespondents, obtain some information on nonrespondents, avoid duplication, avoid failing to contact part of the sample, protect confidentiality of the data, provide anonymity, and maintain a cooperative spirit in the target population. Interviewers should be well trained and coached in regard to how to approach the respondents, how to conduct the interview, how to handle various answers, and how to inform respondents about what is expected of them during the interview.

Step 12: Edit and Code the Data

Editing of data is analogous to editing newspaper copy. The editor's job is to make sure that the text meets certain standards and that errors are corrected. The editor checks for missing data, for inconsistencies, and for problems that can be remedied. Editing of data should be done as soon as possible after data collection.

To permit computerized analysis of data, it is essential that the variables be reduced to a form in which a numerical value may be assigned to each possible choice. This process is referred to as **coding.** It is carried out simultaneously with editing. Coding may be done either by use of an ad hoc coding system specifically developed for your own data base or by use of a standard coding system. A well-accepted technique for the coding of diseases or causes of death is the International Classification of Diseases (World Health Organization, 1977). This flexible system can provide either a broad categorization of disease groups or quite detailed coding of specific entities. For years, the standard procedure was to use punch cards for computer entry. The current method of choice is to enter data directly via an interactive terminal. In this way, a validation program is

able to inform the key-entry operator immediately about possibly invalid data. To maintain accuracy, it is essential that, by program or otherwise, the data entry be verified.

Step 13: Analyze the Data

After data have been collected, edited, coded, and key-entered, they are almost ready for analysis. But a preliminary step is needed: some advance data analysis to ferret out possible outliers, look at the distribution of the various variables, provide an item analysis for the variables of special interest, and assess the amount of missing data. Once this analysis is completed, you are ready to perform the major analysis of data, a task dictated by the specific objectives of the survey.

Step 14: Report the Findings

The report should begin with background information that provides a rationale for the study. It should indicate the specific objectives that the survey seeks to accomplish. A "methods" section should describe the target population, the test instruments, and the sampling design. The "results" section should discuss the findings and possible future implications.

7.2   EVALUATION OF A RESEARCH REPORT

It is quite unlikely that all the users of this book will become regular producers of research literature. But, almost without exception, everyone will be a consumer of such literature. Research literature comes in many forms: books, journal articles, monographs, administrative documents, program evaluations, and the like.

A valuable skill to develop is the ability to critically read and evaluate research literature. Without this, a person is unable to differentiate between a pedestrian report and one of quality. A top-grade report stands unshaken under the critical process of **peer review.** In a sense, every user of literature, by doing a critical analysis, is carrying peer review to its ultimate step.

It is a well-known, if regrettable, fact that some research literature is of poor quality. After wading through a mire of jargon, inconsistencies, poor grammar, tangles of qualifications, and some muddy logic, the user is expected to draw a brilliantly clear scientific conclusion. This problem is chronic in much scientific writing. A full discussion is well beyond the scope of this book. See one of the several excellent treatments of the subject (for example, Flesch, 1974; Sheen, 1982).

A parallel problem exists when dealing with the quantitative aspects of a report. We hope that by reading this section you will gain at least a glimmer of

how to be critical, analytical, and discriminating in your use of research literature.

Researchers, being human, must exercise constant vigilance to avoid bias while working toward a prized objective. As mentioned briefly in Chapter 1, bias may well creep in—usually inadvertently, perhaps subconsciously, and often as a consequence of some aspect of the research design. Although the best researchers are carefully trained in its avoidance, bias assumes so many forms that it is difficult to recognize and avoid them all. By examining a few of these, we should be more capable of effectively evaluating a research report. For a comprehensive catalogue of research bias, see Sackett (1979).

Observer Bias

When the observer (or interviewer) is fully aware that the person being interviewed has a certain disease, the observer may subconsciously attribute certain characteristics to the subject. The result of this **observer bias** is that those characteristics are more likely to be recorded for cases than for controls. The solution of choice is to "blind" the observer as to whether the subject is a case or a control.

Sampling Bias

Bias may enter whenever samples are chosen in a nonrandom fashion. **Convenience sampling** (choosing only subjects who are easy to find) leads almost invariably to biased results. **Systematic sampling** (choosing every nth person from a list) carries the potential of subtle error, especially if the list has some cyclical pattern. Telephone and household sampling have their own potentials for bias. What if no one answers the phone or comes to the door? Should the interviewer skip that household? On the contrary. The interviewer should try again (and again), realizing that a household where no one is at home in the daytime is quite different from one where someone is nearly always present.

Selection Bias

Were the cases and controls drawn from the same population? This question, which sounds simple, has profound implications. **Selection bias** may lead to a false association between a disease and some factor because of different probabilities of selecting persons with and without the disease and with and without the variable of interest. This problem was first quantified by Berkson (1946) and is sometimes called **Berksonian bias,** or hospital selection bias.

Response Bias

When participation in a study is voluntary, **response bias** (sometimes called **nonrespondent bias** or **self-selection bias**) is important. Owing to their psychological makeup, internal motivation, concern for their own health, educa-

tional background, and many other reasons, persons who choose voluntarily to participate are known to differ from those who decline. Nevertheless, many important research studies (e.g., the landmark Framingham Heart Study—the Massachusetts study that reported on the dangers to "yo-yo" dieters, who are shortening their life expectancy by swinging through cycles of weight loss and gain) depend in part on volunteers. A way to control for response bias is to compare characteristics of volunteer subgroups with those of randomly chosen subgroups.

Dropout Bias

Dropout bias is the mirror image of response bias. In long-term studies, a certain proportion of participants, for reasons of their own, choose to drop out. These persons are likely to differ from those who continue.

Memory Bias

There are several well-known aspects of **memory bias** (also known as **subjective bias**). Memory for recent events is much more accurate than for remote events. Hence, persons interviewed concerning past illnesses tend to report a greater prevalence in the recent past than in the distant past (Stocks, 1944). A perhaps more profound form of memory bias is the tendency of persons with a disease to overemphasize the importance of events they may consider to be predisposing causes (e.g., breast cancer patients who trace their disease to traumatic breast injury).

Participant Bias

Participant bias is an interesting form of bias that derives from the participant's knowledge of being a member of the experimental or control group and his or her perception of the research objectives. For example, a member of a heart disease intervention study may report and exaggerate minor symptoms actually unrelated to the disease under study.

Lead-Time Bias

Does early detection of chronic disease actually result in improved survival or does it merely provide a longer period between first detection and death? This fascinating question of **lead-time bias** is fully considered in Cole and Morrison (1980).

Keys to a Systematic Approach

Awareness of the potential for bias underlies a critical reading of any research report. But bias is not the only issue to keep in mind. A great deal may be learned by using a systematic approach toward a critique of any research liter-

ature. Here are some of the most important questions that should be considered:

1. *Research objectives.* Does the research report clearly state its objectives? Do the conclusions address the same objectives?

2. *Study design.* What type of study was it? Was sample selection random and appropriate to the study design? Were cases and controls comparable and drawn from the same reference group?

3. *Data collection.* Were criteria for diagnosis precisely defined? Were end points (or outcome criteria) clearly stated? Were research instruments (whether mechanical or electronic devices, or printed questionnaires) standardized? Can the study be independently replicated?

4. *Discussion of results.* Are results presented clearly and quantitatively? Do tables and figures agree with the text? Are various tables consistent with one another?

5. *Data analysis.* Does the report address the statistical significance of its results? If not, are you able to draw a reasonable inference of significance (or nonsignificance) from the data as presented? Were the statistical tests appropriate to the data? Does the report discuss alternative explanations for what might be spurious statistical significance?

6. *Conclusions.* Are the findings justified by the data? Do the findings relate appropriately to the research objectives originally set forth?

Serious users of research literature have found this step-by-step approach extremely helpful. For a more thorough discussion, see the excellent treatment of this topic in Colton (1974).

Conclusion

Two fundamental research tools, the health survey and the research report, are inseparable parts of the same process: that of aiding scientists, managers, and public officials in their decision making. Health surveys need careful planning; a systematic stepwise procedure is the best means of avoiding error in their use. Research reports are read by nearly everyone in the health sciences. It is important to develop a critical eye to distinguish between ordinary reports and those of quality. Especially when dealing with human populations, the researcher is susceptible to many sources of bias. An understanding of the origins of bias, and of the means to avoid bias in whatever form, helps the user assess the quality of any research report.

Vocabulary List

coding	mailed questionnaire	response bias
consumer	memory bias (subjective	(nonrespondent bias;
convenience	bias)	self-selection bias)
sampling	observer bias	sampling bias
dropout bias	participant bias	selection bias
editing	peer review	(Berksonian bias)
element	person-to-person	systematic sampling
interviewer bias	interview	target population
lead-time bias	pretest	telephone interview

Exercises

17.1 Prepare an outline for a health survey on a subject of special interest to you.

17.2 Locate a completed health survey. Is it constructed in keeping with the guidelines of this chapter? In what ways is it imperfectly planned? What would you do to improve it?

17.3 Choose a scientific article that reports on research in your own field. Subject it to the evaluation process suggested in this chapter.

17.4 Using the survey of Exercise 17.2, the article for Exercise 17.3, or any health survey report, discuss how the authors handled potential bias. What steps did they take to minimize it? What types of bias may have crept in? How could these have been avoided?

Epilogue

We hope that the techniques and methods presented in this book will provide you with some useful tools that can be used to separate fact from fiction, to determine the significance of experimental results, and ultimately to assist your search for truth. The road to truth is seldom an easy one, but a great deal of satisfaction can be attained while traversing it. This is particularly true when one is able to establish the significance of a new finding, to learn that a commonly accepted approach is not really valid, or to gain the kind of insight that begins to shed new light on the process of discovery. The journey may be rough, but it is surely worthwhile. Godspeed!

Appendix A Binomial Probability Table

| | | | | | | | p | | | | | | |
		.01	.05	.10	.15	.20	.25	.30	1/3	.35	.40	.45	.50
n	x	.01	.05	.10	.15	.20	.25	.30	1/3	.35	.40	.45	.50
1	0	.9900	.9500	.9000	.8500	.8000	.7500	.7000	.6667	.6500	.6000	.5500	.5000
	1	.0100	.0500	.1000	.1500	.2000	.2500	.3000	.3333	.3500	.4000	.4500	.5000
2	0	.9801	.9025	.8100	.7225	.6400	.5625	.4900	.4444	.4225	.3600	.3025	.2500
	1	.0198	.0950	.1800	.2550	.3200	.3750	.4200	.4444	.4550	.4800	.4950	.5000
	2	.0001	.0025	.0100	.0225	.0400	.0625	.0900	.1111	.1225	.1600	.2025	.2500
3	0	.9703	.8574	.7290	.6141	.5120	.4219	.3430	.2963	.2746	.2160	.1664	.1250
	1	.0294	.1354	.2430	.3251	.3840	.4219	.4410	.4444	.4436	.4320	.4084	.3750
	2	.0003	.0071	.0270	.0574	.0960	.1406	.1890	.2222	.2389	.2880	.3341	.3750
	3	.0000	.0001	.0010	.0034	.0080	.0156	.0270	.0370	.0429	.0640	.0911	.1250
4	0	.9606	.8145	.6561	.5220	.4096	.3164	.2401	.1975	.1785	.1296	.0915	.0625
	1	.0388	.1715	.2916	.3685	.4096	.4219	.4116	.3951	.3845	.3456	.2995	.2500
	2	.0006	.0135	.0486	.0975	.1536	.2109	.2646	.2963	.3105	.3456	.3675	.3750
	3	.0000	.0005	.0036	.0115	.0256	.0469	.0756	.0988	.1115	.1536	.2005	.2500
	4	.0000	.0000	.0001	.0005	.0016	.0039	.0081	.0123	.0150	.0256	.0410	.0625
5	0	.9510	.7738	.5905	.4437	.3277	.2373	.1681	.1317	.1160	.0778	.0503	.0312
	1	.0480	.2036	.3280	.3915	.4096	.3955	.3601	.3292	.3124	.2592	.2059	.1563
	2	.0010	.0214	.0729	.1382	.2048	.2637	.3087	.3292	.3364	.3456	.3369	.3125
	3	.0000	.0012	.0081	.0244	.0512	.0879	.1323	.1646	.1812	.2304	.2757	.3125
	4	.0000	.0000	.0005	.0021	.0064	.0146	.0284	.0412	.0487	.0768	.1127	.1563
	5	.0000	.0000	.0000	.0001	.0003	.0010	.0024	.0041	.0053	.0102	.0185	.0312
6	0	.9415	.7351	.5314	.3771	.2621	.1780	.1176	.0878	.0754	.0467	.0277	.0156
	1	.0570	.2321	.3543	.3994	.3932	.3559	.3026	.2634	.2437	.1866	.1359	.0938
	2	.0015	.0306	.0984	.1762	.2458	.2967	.3241	.3292	.3280	.3110	.2779	.2344
	3	.0000	.0021	.0146	.0414	.0819	.1318	.1852	.2195	.2355	.2765	.3032	.3125
	4	.0000	.0001	.0012	.0055	.0154	.0330	.0596	.0823	.0951	.1382	.1861	.2344
	5	.0000	.0000	.0001	.0004	.0015	.0044	.0102	.0165	.0205	.0369	.0609	.0938
	6	.0000	.0000	.0000	.0000	.0001	.0002	.0007	.0014	.0018	.0041	.0083	.0156

continued

n	x	.01	.05	.10	.15	.20	.25	.30	1/3	.35	.40	.45	.50
	0	.9321	.6983	.4783	.3206	.2097	.1335	.0824	.0585	.0490	.0280	.0152	.0078
	1	.0659	.2573	.3720	.3960	.3670	.3114	.2470	.2048	.1848	.1306	.0872	.0547
	2	.0020	.0406	.1240	.2096	.2753	.3115	.3177	.3073	.2985	.2613	.2140	.1641
7	3	.0000	.0036	.0230	.0617	.1147	.1730	.2269	.2561	.2679	.2903	.2919	.2734
	4	.0000	.0002	.0025	.0109	.0286	.0577	.0972	.1280	.1442	.1935	.2388	.2734
	5	.0000	.0000	.0002	.0011	.0043	.0116	.0250	.0384	.0466	.0775	.1172	.1641
	6	.0000	.0000	.0000	.0001	.0004	.0012	.0036	.0064	.0084	.0172	.0320	.0547
	7	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0005	.0006	.0016	.0037	.0078
	0	.9227	.6634	.4305	.2725	.1678	.1001	.0576	.0390	.0319	.0168	.0084	.0039
	1	.0746	.2794	.3826	.3847	.3355	.2670	.1977	.1561	.1372	.0896	.0548	.0313
	2	.0026	.0514	.1488	.2376	.2936	.3114	.2965	.2731	.2587	.2090	.1570	.1093
	3	.0001	.0054	.0331	.0838	.1468	.2077	.2541	.2731	.2786	.2787	.2569	.2188
8	4	.0000	.0004	.0046	.0185	.0459	.0865	.1361	.1707	.1875	.2322	.2626	.2734
	5	.0000	.0000	.0004	.0027	.0092	.0231	.0467	.0683	.0808	.1239	.1718	.2188
	6	.0000	.0000	.0000	.0002	.0011	.0038	.0100	.0171	.0217	.0413	.0704	.1093
	7	.0000	.0000	.0000	.0000	.0001	.0004	.0012	.0024	.0034	.0078	.0164	.0313
	8	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0002	.0007	.0017	.0039
	0	.9135	.6302	.3874	.2316	.1342	.0751	.0404	.0260	.0207	.0101	.0046	.0020
	1	.0831	.2986	.3874	.3678	.3020	.2252	.1556	.1171	.1004	.0604	.0339	.0175
	2	.0033	.0628	.1722	.2597	.3020	.3004	.2668	.2341	.2162	.1613	.1110	.0703
	3	.0001	.0078	.0447	.1070	.1762	.2336	.2669	.2731	.2716	.2508	.2119	.1641
9	4	.0000	.0006	.0074	.0283	.0660	.1168	.1715	.2048	.2194	.2508	.2600	.2461
	5	.0000	.0000	.0008	.0050	.0165	.0389	.0735	.1024	.1181	.1672	.2128	.2461
	6	.0000	.0000	.0001	.0006	.0028	.0087	.0210	.0341	.0424	.0744	.1160	.1641
	7	.0000	.0000	.0000	.0000	.0003	.0012	.0039	.0073	.0098	.0212	.0407	.0703
	8	.0000	.0000	.0000	.0000	.0000	.0001	.0004	.0009	.0013	.0035	.0083	.0175
	9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0003	.0008	.0020
	0	.9044	.5987	.3487	.1969	.1074	.0563	.0282	.0173	.0135	.0060	.0025	.0010
	1	.0913	.3152	.3874	.3474	.2684	.1877	.1211	.0867	.0725	.0404	.0208	.0097
	2	.0042	.0746	.1937	.2759	.3020	.2816	.2335	.1951	.1756	.1209	.0763	.0440
	3	.0001	.0105	.0574	.1298	.2013	.2503	.2668	.2601	.2522	.2150	.1664	.1172
	4	.0000	.0009	.0112	.0401	.0881	.1460	.2001	.2276	.2377	.2508	.2384	.2051
10	5	.0000	.0001	.0015	.0085	.0264	.0584	.1030	.1366	.1536	.2007	.2340	.2460
	6	.0000	.0000	.0001	.0013	.0055	.0162	.0367	.0569	.0689	.1114	.1596	.2051
	7			.0000	.0001	.0008	.0031	.0090	.0163	.0212	.0425	.0746	.1172
	8				.0000	.0001	.0004	.0015	.0030	.0043	.0106	.0229	.0440
	9				.0000	.0000	.0000	.0001	.0003	.0005	.0016	.0042	.0097
	10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0010
	0	.8953	.5688	.3138	.1673	.0859	.0422	.0198	.0116	.0088	.0036	.0014	.0005
	1	.0995	.3293	.3836	.3249	.2362	.1549	.0932	.0636	.0518	.0266	.0125	.0054
	2	.0050	.0867	.2130	.2866	.2953	.2581	.1998	.1590	.1395	.0887	.0513	.0268
	3	.0002	.0136	.0711	.1518	.2215	.2581	.2568	.2384	.2255	.1774	.1259	.0806
	4	.0000	.0015	.0157	.0535	.1107	.1721	.2201	.2384	.2427	.2365	.2060	.1611
	5		.0001	.0025	.0132	.0387	.0803	.1321	.1669	.1830	.2207	.2360	.2256
11	6		.0000	.0003	.0024	.0097	.0267	.0566	.0835	.0986	.1471	.1931	.2256
	7			.0000	.0003	.0018	.0064	.0173	.0298	.0379	.0701	.1128	.1611
	8				.0000	.0002	.0011	.0037	.0075	.0102	.0234	.0462	.0806
	9					.0000	.0001	.0006	.0012	.0018	.0052	.0126	.0268
	10						.0000	.0000	.0001	.0002	.0007	.0020	.0054
	11	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0005

								p					
n	*x*	.01	.05	.10	.15	.20	.25	.30	1/3	.35	.40	.45	.50
	0	.8864	.5404	.2824	.1422	.0687	.0317	.0138	.0077	.0057	.0022	.0008	.0002
	1	.1074	.3412	.3766	.3013	.2062	.1267	.0712	.0462	.0367	.0174	.0075	.0030
	2	.0060	.0988	.2301	.2923	.2834	.2323	.1678	.1272	.1089	.0638	.0338	.0161
	3	.0002	.0174	.0853	.1720	.2363	.2581	.2397	.2120	.1954	.1419	.0924	.0537
	4	.0000	.0020	.0213	.0683	.1328	.1936	.2312	.2384	.2366	.2129	.1700	.1208
	5		.0002	.0038	.0193	.0532	.1032	.1585	.1908	.2040	.2270	.2225	.1934
12	6		.0000	.0004	.0039	.0155	.0401	.0792	.1113	.1281	.1766	.2124	.2256
	7			.0001	.0006	.0033	.0115	.0291	.0477	.0591	.1009	.1489	.1934
	8			.0000	.0001	.0005	.0024	.0078	.0149	.0199	.0420	.0761	.1208
	9				.0000	.0001	.0004	.0015	.0033	.0048	.0125	.0277	.0537
	10					.0000	.0000	.0002	.0005	.0007	.0025	.0068	.0161
	11						.0000	.0000	.0000	.0001	.0003	.0010	.0030
	12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002
	0	.8601	.4633	.2059	.0874	.0352	.0134	.0047	.0023	.0016	.0005	.0001	.0000
	1	.1301	.3667	.3431	.2312	.1319	.0668	.0306	.0171	.0126	.0047	.0016	.0005
	2	.0092	.1348	.2669	.2856	.2309	.1559	.0915	.0599	.0475	.0219	.0090	.0032
	3	.0004	.0307	.1285	.2185	.2502	.2252	.1701	.1299	.1110	.0634	.0317	.0139
	4	.0000	.0049	.0429	.1156	.1876	.2252	.2186	.1948	.1792	.1268	.0780	.0416
	5		.0005	.0105	.0449	.1031	.1651	.2061	.2143	.2124	.1859	.1404	.0917
	6		.0001	.0019	.0132	.0430	.0918	.1473	.1786	.1905	.2066	.1914	.1527
	7		.0000	.0003	.0030	.0139	.0393	.0811	.1148	.1320	.1771	.2013	.1964
15	8			.0000	.0005	.0034	.0131	.0348	.0574	.0710	.1181	.1657	.1964
	9				.0001	.0007	.0034	.0115	.0223	.0298	.0612	.1049	.1527
	10				.0000	.0001	.0007	.0030	.0067	.0096	.0245	.0514	.0917
	11					.0000	.0001	.0006	.0015	.0023	.0074	.0192	.0416
	12						.0000	.0001	.0003	.0004	.0016	.0052	.0139
	13							.0000	.0000	.0001	.0003	.0010	.0032
	14								.0000	.0000	.0000	.0001	.0005
	15	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
	0	.8179	.3583	.1216	.0388	.0115	.0032	.0008	.0003	.0002	.0000	.0000	.0000
	1	.1652	.3773	.2701	.1368	.0577	.0211	.0068	.0030	.0019	.0005	.0001	.0000
	2	.0159	.1887	.2852	.2293	.1369	.0670	.0279	.0143	.0100	.0031	.0008	.0002
	3	.0010	.0596	.1901	.2428	.2053	.1339	.0716	.0429	.0323	.0124	.0040	.0011
	4	.0000	.0133	.0898	.1821	.2182	.1896	.1304	.0911	.0738	.0350	.0140	.0046
	5		.0023	.0319	.1029	.1746	.2024	.1789	.1457	.1272	.0746	.0364	.0148
	6		.0003	.0089	.0454	.1091	.1686	.1916	.1821	.1714	.1244	.0746	.0370
	7		.0000	.0020	.0160	.0546	.1124	.1643	.1821	.1844	.1659	.1221	.0739
	8			.0003	.0046	.0221	.0609	.1144	.1480	.1614	.1797	.1623	.1201
	9			.0001	.0011	.0074	.0270	.0653	.0987	.1158	.1597	.1771	.1602
20	10			.0000	.0002	.0020	.0100	.0309	.0543	.0686	.1172	.1593	.1762
	11				.0000	.0005	.0030	.0120	.0247	.0336	.0710	.1185	.1602
	12					.0001	.0007	.0038	.0092	.0136	.0355	.0728	.1201
	13					.0000	.0002	.0010	.0028	.0045	.0145	.0366	.0739
	14						.0000	.0003	.0007	.0012	.0049	.0150	.0370
	15							.0000	.0001	.0003	.0013	.0049	.0148
	16								.0000	.0000	.0003	.0012	.0046
	17										.0000	.0003	.0011
	18											.0000	.0002
	19											.0000	.0000
	20	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

	$F_{.95}$ (use with $\alpha = .05$)								
					df_b				
df_w	1	2	3	4	5	6	7	8	9
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88

continued

					df_b					
					$F_{.95}$ (use with $\alpha = .05$)					
df_w	10	12	15	20	24	30	40	60	120	∞
1	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	255.3	254.30
2	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.19	2.13
15	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

$F_{.99}$ (use with $\alpha = .01$)

df_w	df_b								
	1	2	3	4	5	6	7	8	9
1	4052.00	4999.50	5403.00	5625.00	5764.00	5859.00	5928.00	5981.00	6022.00
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.55
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16
6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98
7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94
11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
24	7.82	5.61	4.79	4.22	3.90	3.67	3.50	3.36	3.26
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18
27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15
28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07
40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41

$F_{.99}$ (use with $\alpha = .01$)

df$_w$	10	12	15	20	df$_b$ 24	30	40	60	120	∞
1	6056.00	6106.00	6157.00	6209.00	6235.00	6261.00	6287.00	6313.00	6339.00	6366.00
2	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	27.23	27.05	26.87	26.69	26.60	26.50	26.41	26.32	26.22	26.13
4	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
5	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
6	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
7	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
8	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.86
9	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
10	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
11	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
12	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
13	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
14	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
15	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
16	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
17	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65
18	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
19	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.49
20	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
21	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
22	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
23	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
24	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
25	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
26	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
27	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
28	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
29	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
30	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
40	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
60	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
120	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
∞	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

Appendix C Percentage Points of the Studentized Range for 2 Through 20 Treatments

				Upper 5% Points					
					k				
df_w	2	3	4	5	6	7	8	9	10
1	17.97	26.98	32.82	37.08	40.41	43.12	45.40	47.36	49.07
2	6.08	8.33	9.80	10.88	11.74	12.44	13.03	13.54	13.99
3	4.50	5.91	6.82	7.50	8.04	8.48	8.85	9.18	9.46
4	3.93	5.04	5.76	6.29	6.71	7.05	7.35	7.60	7.83
5	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99
6	3.46	4.34	4.90	5.30	5.63	5.90	6.12	6.32	6.49
7	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16
8	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92
9	3.20	3.95	4.41	4.76	5.02	5.24	5.43	5.59	5.74
10	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60
11	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.49
12	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.39
13	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32
14	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25
15	3.01	3.67	4.08	4.37	4.59	4.78	4.94	5.08	5.20
16	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15
17	2.98	3.63	4.02	4.30	4.52	4.70	4.86	4.99	5.11
18	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07
19	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04
20	2.95	3.58	3.96	4.23	4.45	4.62	4.77	4.90	5.01
24	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92
30	2.89	3.49	3.85	4.10	4.30	4.46	4.60	4.72	4.82
40	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.73
60	2.83	3.40	3.74	3.98	4.16	4.31	4.44	4.55	4.65
120	2.80	3.36	3.68	3.92	4.10	4.24	4.36	4.47	4.56
∞	2.77	3.31	3.63	3.86	4.03	4.17	4.29	4.39	4.47

SOURCE: From Table 29 of Pearson, E. S., and Hartley, H. O. (1966) *Biometrika: Tables for Statisticians,* Volume I, Third Edition, published by Cambridge University Press.

Upper 5% Points

df$_w$	11	12	13	14	15	16	17	18	19	20
					k					
1	50.59	51.96	53.20	54.33	55.36	56.32	57.22	58.04	58.83	59.56
2	14.39	14.75	15.08	15.38	15.65	15.91	16.14	16.37	16.57	16.77
3	9.72	9.95	10.15	10.35	10.52	10.69	10.84	10.98	11.11	11.24
4	8.03	8.21	8.37	8.52	8.66	8.79	8.91	9.03	9.13	9.23
5	7.17	7.32	7.47	7.60	7.72	7.83	7.93	8.03	8.12	8.21
6	6.65	6.79	6.92	7.03	7.14	7.24	7.34	7.43	7.51	7.59
7	6.30	6.43	6.55	6.66	6.76	6.85	6.94	7.02	7.10	7.17
8	6.05	6.18	6.29	6.39	6.48	6.57	6.65	6.73	6.80	6.87
9	5.87	5.98	6.09	6.19	6.28	6.36	6.44	6.51	6.58	6.64
10	5.72	5.83	5.93	6.03	6.11	6.19	6.27	6.34	6.40	6.47
11	5.61	5.71	5.81	5.90	5.98	6.06	6.13	6.20	6.27	6.33
12	5.51	5.61	5.71	5.80	5.88	5.95	6.02	6.09	6.15	6.21
13	5.43	5.53	5.63	5.71	5.79	5.86	5.93	5.99	6.05	6.11
14	5.36	5.46	5.55	5.64	5.71	5.79	5.85	5.91	5.97	6.03
15	5.31	5.40	5.49	5.57	5.65	5.72	5.78	5.85	5.90	5.96
16	5.26	5.35	5.44	5.52	5.59	5.66	5.73	5.79	5.84	5.90
17	5.21	5.31	5.39	5.47	5.54	5.61	5.67	5.73	5.79	5.84
18	5.17	5.27	5.35	5.43	5.50	5.57	5.63	5.69	5.74	5.79
19	5.14	5.23	5.31	5.39	5.46	5.53	5.59	5.65	5.70	5.75
20	5.11	5.20	5.28	5.36	5.43	5.49	5.55	5.61	5.66	5.71
24	5.01	5.10	5.18	5.25	5.32	5.38	5.44	5.49	5.55	5.59
30	4.92	5.00	5.08	5.15	5.21	5.27	5.33	5.38	5.43	5.47
40	4.82	4.90	4.98	5.04	5.11	5.16	5.22	5.27	5.31	5.36
60	4.73	4.81	4.88	4.94	5.00	5.06	5.11	5.15	5.20	5.24
120	4.64	4.71	4.78	4.84	4.90	4.95	5.00	5.04	5.09	5.13
∞	4.55	4.62	4.68	4.74	4.80	4.85	4.89	4.93	4.97	5.01

continued

Upper 1% Points

df$_w$	2	3	4	5	k 6	7	8	9	10
1	90.03	135.00	164.30	185.60	202.20	215.80	227.20	237.00	245.60
2	14.04	19.02	22.29	24.72	26.63	28.20	29.53	30.68	31.69
3	8.26	10.62	12.17	13.33	14.24	15.00	15.64	16.20	16.69
4	6.51	8.12	9.17	9.96	10.58	11.10	11.55	11.93	12.27
5	5.70	6.98	7.80	8.42	8.91	9.32	9.67	9.97	10.24
6	5.24	6.33	7.03	7.56	7.97	8.32	8.61	8.87	9.10
7	4.95	5.92	6.54	7.01	7.37	7.68	7.94	8.17	8.37
8	4.75	5.64	6.20	6.62	6.96	7.24	7.47	7.68	7.86
9	4.60	5.43	5.96	6.35	6.66	6.91	7.13	7.33	7.49
10	4.48	5.27	5.77	6.14	6.43	6.67	6.87	7.05	7.21
11	4.39	5.15	5.62	5.97	6.25	6.48	6.67	6.84	6.99
12	4.32	5.05	5.50	5.84	6.10	6.32	6.51	6.67	6.81
13	4.26	4.96	5.40	5.73	5.98	6.19	6.37	6.53	6.67
14	4.21	4.89	5.32	5.63	5.88	6.08	6.26	6.41	6.54
15	4.17	4.84	5.25	5.56	5.80	5.99	6.16	6.31	6.44
16	4.13	4.79	5.19	5.49	5.72	5.92	6.08	6.22	6.35
17	4.10	4.74	5.14	5.43	5.66	5.85	6.01	6.15	6.27
18	4.07	4.70	5.09	5.38	5.60	5.79	5.94	6.08	6.20
19	4.05	4.67	5.05	5.33	5.55	5.73	5.89	6.02	6.14
20	4.02	4.64	5.02	5.29	5.51	5.69	5.84	5.97	6.09
24	3.96	4.55	4.91	5.17	5.37	5.54	5.69	5.81	5.92
30	3.89	4.45	4.80	5.05	5.24	5.40	5.54	5.65	5.76
40	3.82	4.37	4.70	4.93	5.11	5.26	5.39	5.50	5.60
60	3.76	4.28	4.59	4.82	4.99	5.13	5.25	5.36	5.45
120	3.70	4.20	4.50	4.71	4.87	5.01	5.12	5.21	5.30
∞	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16

					Upper 1% Points.					
					k					
df_w	11	12	13	14	15	16	17	18	19	20
1	253.2	260.0	266.2	271.8	277.0	281.8	286.3	290.4	294.3	298.0
2	32.59	33.40	34.13	34.81	35.43	36.00	36.53	37.03	37.50	37.95
3	17.13	17.53	17.89	18.22	18.52	18.81	19.07	19.32	19.55	19.77
4	12.57	12.84	13.09	13.32	13.53	13.73	13.91	14.08	14.24	14.40
5	10.48	10.70	10.89	11.08	11.24	11.40	11.55	11.68	11.81	11.93
6	9.30	9.48	9.65	9.81	9.95	10.08	10.21	10.32	10.43	10.54
7	8.55	8.71	8.86	9.00	9.12	9.24	9.35	9.46	9.55	9.65
8	8.03	8.18	8.31	8.44	8.55	8.66	8.76	8.85	8.94	9.03
9	7.65	7.78	7.91	8.03	8.13	8.23	8.33	8.41	8.49	8.57
10	7.36	7.49	7.60	7.71	7.81	7.91	7.99	8.08	8.15	8.23
11	7.13	7.25	7.36	7.46	7.56	7.65	7.73	7.81	7.88	7.95
12	6.94	7.06	7.17	7.26	7.36	7.44	7.52	7.59	7.66	7.73
13	6.79	6.90	7.01	7.10	7.19	7.27	7.35	7.42	7.48	7.55
14	6.66	6.77	6.87	6.96	7.05	7.13	7.20	7.27	7.33	7.39
15	6.55	6.66	6.76	6.84	6.93	7.00	7.07	7.14	7.20	7.26
16	6.46	6.56	6.66	6.74	6.82	6.90	6.97	7.03	7.09	7.15
17	6.38	6.48	6.57	6.66	6.73	6.81	6.87	6.94	7.00	7.05
18	6.31	6.41	6.50	6.58	6.65	6.73	6.79	6.85	6.91	6.97
19	6.25	6.34	6.43	6.51	6.58	6.65	6.72	6.78	6.84	6.89
20	6.19	6.28	6.37	6.45	6.52	6.59	6.65	6.71	6.77	6.82
24	6.02	6.11	6.19	6.26	6.33	6.39	6.45	6.51	6.56	6.61
30	5.85	5.93	6.01	6.08	6.14	6.20	6.26	6.31	6.36	6.41
40	5.69	5.76	5.83	5.90	5.96	6.02	6.07	6.12	6.16	6.21
60	5.53	5.60	5.67	5.73	5.78	5.84	5.89	5.93	5.97	6.01
120	5.37	5.44	5.50	5.56	5.61	5.66	5.71	5.75	5.79	5.83
∞	5.23	5.29	5.35	5.40	5.45	5.49	5.54	5.57	5.61	5.65

D Critical Values of *n* for the Sign Test

In the body of the table, the first number of the pair usually refers to the positive values, and the second number to the negative values.

n	α(two-sided) 0.10 α(one-sided) 0.05	0.05 0.025	0.02 0.01	0.01 0.005
1				
2				
3				
4				
5	0, 5			
6	0, 6	0, 6		
7	0, 7	0, 7	0, 7	
8	1, 7	0, 8	0, 8	0, 8
9	1, 8	1, 8	0, 9	0, 9
10	1, 9	1, 9	0, 10	0, 10
11	2, 9	1, 10	1, 10	0, 11
12	2, 10	2, 10	1, 11	1, 11
13	3, 10	2, 11	1, 12	1, 12
14	3, 11	2, 12	2, 12	1, 13
15	3, 12	3, 12	2, 13	2, 13
16	4, 12	3, 13	2, 14	2, 14
17	4, 13	4, 13	3, 14	2, 15
18	5, 13	4, 14	3, 15	3, 15
19	5, 14	4, 15	4, 15	3, 16
20	5, 15	5, 15	4, 16	3, 17
21	6, 15	5, 16	4, 17	4, 17
22	6, 16	5, 17	5, 17	4, 18
23	7, 16	6, 17	5, 18	4, 19
24	7, 17	6, 18	5, 19	5, 19
25	7, 18	7, 18	6, 19	5, 20
26	8, 18	7, 19	6, 20	6, 20
27	8, 19	7, 20	7, 20	6, 21
28	9, 19	8, 20	7, 21	6, 22
29	9, 20	8, 21	7, 22	7, 22
30	10, 20	9, 21	8, 22	7, 23
31	10, 21	9, 22	8, 23	7, 24
32	10, 22	9, 23	8, 24	8, 24
33	11, 22	10, 23	9, 24	8, 25
34	11, 23	10, 24	9, 25	9, 25
35	12, 23	11, 24	10, 25	9, 26

n	α(two-sided) 0.10 α(one-sided) 0.05	0.05 0.025	0.02 0.01	0.01 0.005
36	12, 24	11, 25	10, 26	9, 27
37	13, 24	12, 25	10, 27	10, 27
38	13, 25	12, 26	11, 27	10, 28
39	13, 26	12, 27	11, 28	11, 28
40	14, 26	13, 27	12, 28	11, 29
41	14, 27	13, 28	12, 29	11, 30
42	15, 27	14, 28	13, 29	12, 30
43	15, 28	14, 29	13, 30	12, 31
44	16, 28	15, 29	13, 31	13, 31
45	16, 29	15, 30	14, 31	13, 32
46	16, 30	15, 31	14, 32	13, 33
47	17, 30	16, 31	15, 32	14, 33
48	17, 31	16, 32	15, 33	14, 34
49	18, 31	17, 32	15, 34	15, 34
50	18, 32	17, 33	16, 34	15, 35
51	19, 32	18, 33	16, 35	15, 36
52	19, 33	18, 34	17, 35	16, 36
53	20, 33	18, 35	17, 36	16, 37
54	20, 34	19, 35	18, 36	17, 37
55	20, 35	19, 36	18, 37	17, 38
56	21, 35	20, 36	18, 38	17, 39
57	21, 36	20, 37	19, 38	18, 39
58	22, 36	21, 37	19, 39	18, 40
59	22, 37	21, 38	20, 39	19, 40
60	23, 37	21, 39	20, 40	19, 41

Row number										
00000	10097	32533	76520	13586	34673	54876	80959	09177	39292	74945
00001	37542	04805	64894	74296	24805	24037	20636	10402	00822	91665
00002	08422	68953	19645	09303	23209	02560	15953	34764	35080	33606
00003	99019	02529	09376	70715	38311	31165	88676	74397	04436	27659
00004	12807	99970	80157	36147	64032	36653	98951	16877	12171	76833
00005	66065	74717	34072	76850	36697	36170	65813	39885	11199	29170
00006	31060	10805	45571	82406	35303	42614	86799	07439	23403	09732
00007	85269	77602	02051	65692	68665	74818	73053	85247	18623	88579
00008	63573	32135	05325	47048	90553	57548	28468	28709	83491	25624
00009	73796	45753	03529	64778	35808	34282	60935	20344	35273	88435
00010	98520	17767	14905	68607	22109	40558	60970	93433	50500	73998
00011	11805	05431	39808	27732	50725	68248	29405	24201	52775	67851
00012	83452	99634	06288	98033	13746	70078	18475	40610	68711	77817
00013	88685	40200	86507	58401	36766	67951	90364	76493	29609	11062
00014	99594	67348	87517	64969	91826	08928	93785	61368	23478	34113
00015	65481	17674	17468	50950	58047	76974	73039	57186	40218	16544
00016	80124	35635	17727	08015	45318	22374	21115	78253	14385	53763
00017	74350	99817	77402	77214	43236	00210	45521	64237	96286	02655
00018	69916	26803	66252	29148	36936	87203	76621	13990	94400	56418
00019	09893	20505	14225	68514	46427	56788	96297	78822	54382	14598
00020	91499	14523	68479	27686	46162	83554	94750	89923	37089	20048
00021	80336	94598	26940	36858	70297	34135	53140	33340	42050	82341
00022	44104	81949	85157	47954	32979	26575	57600	40881	22222	06413
00023	12550	73742	11100	02040	12860	74697	96644	89439	28707	25815
00024	63606	49329	16505	34484	40219	52563	43651	77082	07207	31790
00025	61196	90446	26457	47774	51924	33729	65394	59593	42582	60527
00026	15474	45266	95270	79953	59367	83848	82396	10118	33211	59466
00027	94557	28573	67897	54387	54622	44431	91190	42592	92927	45973
00028	42481	16213	97344	08721	16868	48767	03071	12059	25701	46670
00029	23523	78317	73208	89837	68935	91416	26252	29663	05522	82562
00030	04493	52494	75246	33824	45862	51025	61962	79335	65337	12472
00031	00549	97654	64051	88159	96119	63896	54692	82391	23287	29529
00032	35963	15307	26898	09354	33351	35462	77974	50024	90103	39333
00033	59808	08391	45427	26842	83609	49700	13021	24892	78565	20106
00034	46058	85236	01390	92286	77281	44077	93910	83647	70617	42941
00035	32179	00597	87379	25241	05567	07007	86743	17157	85394	11838
00036	69234	61406	20117	45204	15956	60000	18743	92423	97118	96338
00037	19565	41430	01758	75379	40419	21585	66674	36806	84962	85207
00038	45155	14938	19476	07246	43667	94543	59047	90033	20826	69541
00039	94864	31994	36168	10851	34888	81553	01540	35456	05014	51176

Row number										
00040	98086	24826	45240	28404	44999	08896	39094	73407	35441	31880
00041	33185	16232	41941	50949	89435	48581	88695	41994	37548	73043
00042	80951	00406	96382	70774	20151	23387	25016	25298	94624	61171
00043	79752	49140	71961	28296	69861	02591	74852	20539	00387	59579
00044	18633	32537	98145	06571	31010	24674	05455	61427	77938	91936
00045	74029	43902	77557	32270	97790	17119	52527	58021	80814	51748
00046	54178	45611	80993	37143	05335	12969	56127	19255	36040	90324
00047	11664	49883	52079	84827	59381	71539	09973	33440	88461	23356
00048	48324	77928	31249	64710	02295	36870	32307	57546	15020	09994
00049	69074	94138	87637	91976	35584	04401	10518	21615	01848	76938
00050	09188	20097	32825	39527	04220	86304	83389	87374	64278	58044
00051	90045	85497	51981	50654	94938	81997	91870	76150	68476	64659
00052	73189	50207	47677	26269	62290	64464	27124	67018	41361	82760
00053	75768	76490	20971	87749	90429	12272	95375	05871	93823	43178
00054	54016	44056	66281	31003	00682	27398	20714	53295	07706	17813
00055	08358	69910	78542	42785	13661	58873	04618	97553	31223	08420
00056	28306	03264	81333	10591	40510	07893	32604	60475	94119	01840
00057	53840	86233	81594	13628	51215	90290	28466	68795	77762	20791
00058	91757	53741	61613	62669	50263	90212	55781	76514	83483	47055
00059	89415	92694	00397	58391	12607	17646	48949	72306	94541	37408
00060	77513	03820	86864	29901	68414	82774	51908	13980	72893	55507
00061	19502	37174	69979	20288	55210	29773	74287	75251	65344	67415
00062	21818	59313	93278	81757	05686	73156	07082	85046	31853	38452
00063	51474	66499	68107	23621	94049	91345	42836	09191	08007	45449
00064	99559	68331	62535	24170	69777	12830	74819	78142	43860	72834
00065	33713	48007	93584	72869	51926	64721	58303	29822	93174	93972
00066	85274	86893	11303	22970	28834	34137	73515	90400	71148	43643
00067	84133	89640	44035	52166	73852	70091	61222	60561	62327	18423
00068	56732	16234	17395	96131	10123	91622	85496	57560	81604	18880
00069	65138	56806	87648	85261	34313	65861	45875	21069	85644	47277
00070	38001	02176	81719	11711	71602	92937	74219	64049	65584	49698
00071	37402	96397	01304	77586	56271	10086	47324	62605	40030	37438
00072	97125	40348	87083	31417	21815	39250	75237	62047	15501	29578
00073	21826	41134	47143	34072	64638	85902	49139	06441	03856	54552
00074	73135	42742	95719	09035	85794	74296	08789	88156	64691	19202
00075	07638	77929	03061	18072	96207	44156	23821	99538	04713	66994
00076	60528	83441	07954	19814	59175	20695	05533	52139	61212	06455
00077	83596	35655	06958	92983	05128	09719	77433	53783	92301	50498
00078	10850	62746	99599	10507	13499	06319	53075	71839	06410	19362
00079	39820	98952	43622	63147	64421	80814	43800	09351	31024	73167
00080	59580	06478	75569	78800	88835	54486	23768	06156	04111	08408
00081	38508	07341	23793	48763	90822	97022	17719	04207	95954	49953
00082	30692	70668	94688	16127	56196	80091	82067	63400	05462	69200
00083	65443	95659	18238	27437	49632	24041	08337	65676	96299	90836
00084	27267	50264	13192	72294	07477	44606	17985	48911	97341	30358
00085	91307	06991	19072	24210	36699	53728	28825	35793	28976	66252
00086	68434	94688	84473	13622	62126	98408	12843	82590	09815	93146
00087	48908	15877	54745	24591	35700	04754	83824	52692	54130	55160
00088	06913	45197	42672	78601	11883	09528	63011	98901	14974	40344
00089	10455	16019	14210	33712	91342	37821	88325	80851	43667	70883

Row number										
00090	12883	97343	65027	61184	04285	01392	17974	15077	90712	26769
00091	21778	30976	38807	36961	31649	42096	63281	02023	08816	47449
00092	19523	59515	65122	59659	86283	68258	69572	13798	16435	91529
00093	67245	52670	35583	16563	79246	86686	76463	34222	26655	90802
00094	60584	47377	07500	37992	45134	26529	26760	83637	41326	44344
00095	53853	41377	36066	94850	58838	73859	49364	73331	96240	43642
00096	24637	38736	74384	89342	52623	07992	12369	18601	03742	83873
00097	83080	12451	38992	22815	07759	51777	97377	27585	51972	37867
00098	16444	24334	36151	99073	27493	70939	85130	32552	54846	54759
00099	60790	18157	57178	65762	11161	78576	45819	52979	65130	04860
00100	03991	10461	93716	16894	66083	24653	84609	58232	88618	19161
00101	38555	95554	32886	59780	08355	60860	29735	47762	71299	23853
00102	17546	73704	92052	46215	55121	29281	59076	07936	27954	58909
00103	32643	52861	95819	06831	00911	98936	76355	93779	80863	00514
00104	69572	68777	39510	35905	14060	40619	29549	69616	33564	60780
00105	24122	66591	27699	06494	14845	46672	61958	77100	90899	75754
00106	61196	30231	92962	61773	41839	55382	17267	70943	78038	70267
00107	30532	21704	10274	12202	39685	23309	10061	68829	55986	66485
00108	03788	97599	75867	20717	74416	53166	35208	33374	87539	08823
00109	48228	63379	85783	47619	53152	67433	35663	52972	16818	60311
00110	60365	94653	35075	33949	42614	29297	01918	28316	98953	73231
00111	83799	42402	56623	34442	34994	41374	70071	14736	09958	18065
00112	32960	07405	36409	83232	99385	41600	11133	07586	15917	06253
00113	19322	53845	57620	52606	66497	68646	78138	66559	19640	99413
00114	11220	94747	07399	37408	48509	23929	27482	45476	85244	35159
00115	31751	57260	68980	05339	15470	48355	88651	22596	03152	19121
00116	88492	99382	14454	04504	20094	98977	74843	93413	22109	78508
00117	30934	47744	07481	83828	73788	06533	28597	20405	94205	20380
00118	22888	48893	27499	98748	60530	45128	74022	84617	82037	10268
00119	78212	16993	35902	91386	44372	15486	65741	14014	87481	37220

Appendix F Table of Probabilities for the Kruskal–Wallis One-Way ANOVA by Ranks*

Sample Sizes					Sample Sizes				
n_1	n_2	n_3	H	p	n_1	n_2	n_3	H	p
2	1	1	2.7000	.500	4	3	2	6.4444	.008
2	2	1	3.6000	.200				6.3000	.011
								5.4444	.046
2	2	2	4.5714	.067				5.4000	.051
			3.7143	.200				4.5111	.098
3	1	1	3.2000	.300					
					4	3	3	6.7455	.010
3	2	1	4.2857	.100				6.7091	.013
			3.8571	.133				5.7909	.046
								5.7273	.050
3	2	2	5.3572	.029				4.7091	.092
			4.7143	.048				4.7000	.101
			4.5000	.067					
			4.4643	.105	4	4	1	6.6667	.010
								6.1667	.022
3	3	1	5.1429	.043				4.9667	.048
			4.5714	.100				4.8667	.054
			4.0000	.129				4.1667	.082
								4.0667	.102
3	3	2	6.2500	.011					
			5.3611	.032					
			5.1389	.061	4	4	2	7.0364	.006
			4.5556	.100				6.8727	.011
			4.2500	.121				5.4545	.046
								5.2364	.052
3	3	3	7.2000	.004				4.5545	.098
			6.4889	.011				4.4455	.103
			5.6889	.029					
			5.6000	.050	4	4	3	7.1439	.010
			5.0667	.086				7.1364	.011
			4.6222	.100				5.5985	.049
								5.5758	.051
4	1	1	3.5714	.200				4.5455	.099
								4.4773	.102
4	2	1	4.8214	.057					
			4.5000	.076	4	4	4	7.6538	.008
			4.0179	.114				7.5385	.011
								5.6923	.049

continued

Sample Sizes						Sample Sizes				
n_1	n_2	n_3	H	p		n_1	n_2	n_3	H	p
4	2	2	6.0000	.014					5.6538	.054
			5.3333	.033					4.6539	.097
			5.1250	.052					4.5001	.104
			4.4583	.100						
			4.1667	.105		5	1	1	3.8571	.143
4	3	1	5.8333	.021		5	2	1	5.2500	.036
			5.2083	.050					5.0000	.048
			5.0000	.057					4.4500	.071
			4.0556	.093					4.2000	.095
			3.8889	.129					4.0500	.119
5	2	2	6.5333	.008					5.6308	.050
			6.1333	.013					4.5487	.099
			5.1600	.034					4.5231	.103
			5.0400	.056						
			4.3733	.090		5	4	4	7.7604	.009
			4.2933	.122					7.7440	.011
									5.6571	.049
5	3	1	6.4000	.012					5.6176	.050
			4.9600	.048					4.6187	.100
			4.8711	.052					4.5527	.102
			4.0178	.095		5	5	1	7.3091	.009
			3.8400	.123					6.8364	.011
									5.1273	.046
5	3	2	6.9091	.009					4.9091	.053
			6.8218	.010					4.1091	.086
			5.2509	.049					4.0364	.105
			5.1055	.052						
			4.6509	.091		5	5	2	7.3385	.010
			4.4945	.101					7.2692	.010
									5.3385	.047
5	3	3	7.0788	.009					5.2462	.051
			6.9818	.011					4.6231	.097
			5.6485	.049					4.5077	.100
			5.5152	.051						
			4.5333	.097		5	5	3	7.5780	.010
			4.4121	.109					7.5429	.010
									5.7055	.046
5	4	1	6.9545	.008					5.6264	.051
			6.8400	.011					4.5451	.100
			4.9855	.044					4.5363	.102
			4.8600	.056						
			3.9873	.098		5	5	4	7.8229	.010
			3.9600	.102					7.7914	.010
									5.6657	.049
5	4	2	7.2045	.009					5.6429	.050
			7.1182	.010					4.5229	.099
			5.2727	.049					4.5200	.101
			5.2682	.050						
			4.5409	.098		5	5	5	8.0000	.009
			4.5182	.101					7.9800	.010
									5.7800	.049
5	4	3	7.4449	.010					5.6600	.051
			7.3949	.011					4.5600	.100
			5.6564	.049					4.5000	.102

Answers to Selected Exercises

Chapter 2

2.1 Simple random sample

2.2 Stratified random sampling

2.3 a. Systematic sampling

      b. Yes, if the variable you are sampling has periodic variation

2.4 a. 7683 persons enrolled in the Honolulu Heart Study, 1969

      c. Statistic

      d. Parameter

2.5 a. A *parameter* is a characteristic of a population and a *statistic* is a characteristic of a sample.

      c. A *simple random sample* is one in which each member of the population has had an equal chance of being selected. This is usually done with the aid of a random number table.

A *convenience sample* is one in which a selected number has not been given an equal chance of being selected. The selected members are included because of some characteristic other than a chance mechanism.

2.7 a. The population is the entire list of 83 individuals with their blood pressure readings. The sample of 10 is those selected by use of the random number table.

      b. The population is the entire list of 7683 individuals in the Honolulu Heart Study Population. Data for a sample of 100 individuals are shown in Table 3.1.

Chapter 3

3.1

	a.		b.	
	Education	qualitative	Weight	continuous
	Weight	quantitative	Height	continuous
	Height	quantitative	Blood glucose	continuous
	Smoking	qualitative	Serum cholesterol	continuous
	Physical activity	qualitative	Systolic blood pressure	continuous
	Blood glucose	quantitative		
	Serum cholesterol	quantitative	Ponderal index	continuous
	Systolic blood pressure	quantitative	Age	continuous
	Ponderal index	quantitative		
	Age	quantitative		

    c. Education   bar chart or pie chart

Education	bar chart or pie chart
Weight	frequency polygon or ogive
Height	frequency polygon or ogive
Smoking	bar chart or pie chart
Physical activity	bar chart or pie chart
Blood glucose	frequency polygon or ogive
Serum cholesterol	frequency polygon or ogive
Systolic blood pressure	frequency polygon or ogive
Ponderal index	frequency polygon or ogive
Age	frequency polygon or ogive

3.2

Diastolic blood pressure	quantitative continuous
Sex	qualitative
Diet status	qualitative

3.3 The shape is approximately symmetrical. The distribution of smokers and non-smokers would not be similar. Smokers' distribution would have a slight positive skew.

3.4 Extreme values are to the left in a negatively skewed distribution and to the right in a positively skewed one.

3.8
a. Bar graph
b. Frequency polygon
c. Pie chart
d. Line graph

3.9 Stem-and-leaf display:

		Frequency
40–49	7 9	2
50–59	0 1 2 2 2 2 3 3 5 5 5 5 5 6 6 6 6 7 7 8 8 8 9 9 9 9 9 9 9 9	30
60–69	0 0 0 0 0 1 1 1 1 1 1 1 1 1 1 2 2 2 3 4 4 5 5 5 6 6 6 6 6 6 6 6 6 7 7 8 8 8 8 8 8	41
70–79	0 0 0 0 0 0 1 1 1 3 3 3 3 3 3 5 5 5 7 7 7 8	21
80–89	0 0 2 3 6	5
90–99	1	1
		Total 100

a. The smallest is 47, largest is 91.
b. 61

3.11

		Frequency
150–154	0 2 2 2 2 2 2 4 4	9
155–159	5 5 5 5 5 5 5 5 5 7 7 7 7 7 7 7 7 9 9 9 9 9	22
160–164	0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 2 2 2 2 2 2 2 2 2 4 4 4	31
165–169	5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 5 6 6 6 7 7 9	24
170–174	0 0 0 0 0 0 0 0 0 1 1 2 2 3	13
175–179	5	1
		Total 100

a. The smallest height is 150 and the largest is 175.
b. The most frequent is 165.
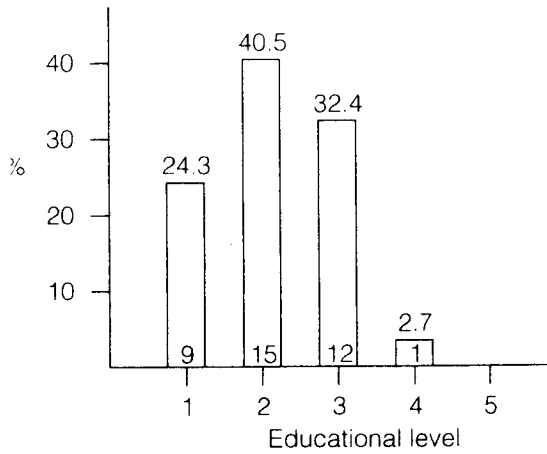
3.13   a. Smokers ($n = 37$)   b. Nonsmokers ($n = 63$)
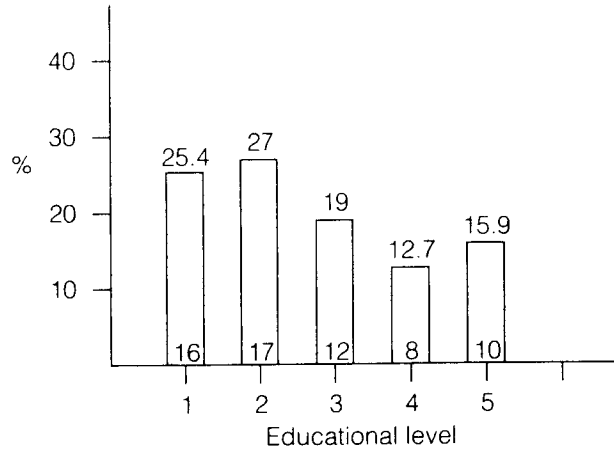
c. There is a higher proportion of nonsmokers with a high school (#4) and technical school (#5) education level.

3.15   Pie chart
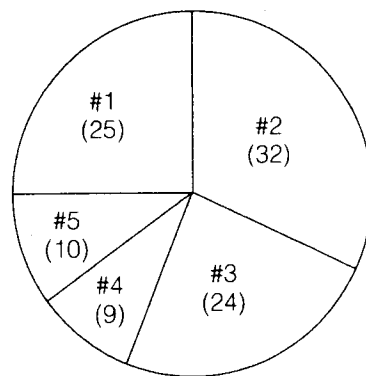
3.17   a. Frequency Table of Weight Loss (in pounds) of 25 Individuals Enrolled in a Weight-Control Program.

Weight Loss (lb)	f	%
1–3	3	12
4–6	6	24
7–9	9	36
10–12	7	28
13–15	0	0
	$\overline{25}$	$\overline{100}$

b and c.
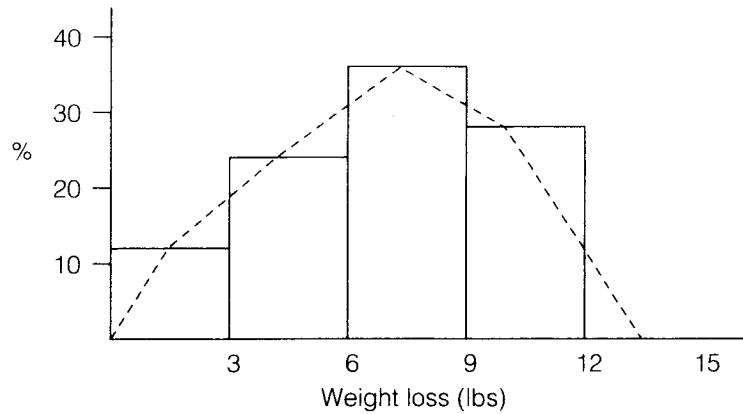
d. The distribution appears to be negatively skewed. A possible interpretation is that there are more individuals with small losses than large losses.
e. The most common weight loss was 9 lb.

Chapter 4

4.1 Mean $= \dfrac{\Sigma n}{n} = \dfrac{24}{6} = 4$

Median $= 4$

Mode $= 5$

Range $= 8 - 1 = 7$

Variance $= \dfrac{\Sigma(x - \bar{x})^2}{n - 1} = \dfrac{32}{5} = 6.40$

Standard deviation $= \sqrt{\text{variance}} = 2.53$

4.3 Range $= 102 - 40 = 62$; median $= 72$; mode $= 70$

4.7 a. $CV_H = \dfrac{100s}{\bar{x}} = \dfrac{100(5.60)}{161.75} = 3.46$

$CV_W = \dfrac{100s}{\bar{x}} = \dfrac{100(8.61)}{64.22} = 13.41$

b. Weight is approximately four times larger.

4.8 a. $\bar{x} = \dfrac{13,010}{100} = 130.10$

$s^2 = \dfrac{1,737,124 - 1,692,601}{99} = 449.73$ $s = 21.21$

b. 108.89, 151.31
c. 87.68, 172.52
d. 66.47, 193.73
e. 68.3%, 95.4%, 99.7%

4.10 Variance $= s^2 = (38.82)^2 = 1506.99$

4.14 a. i. 138,190,128,152,134,108,118,138,108,126,176,112,92,152,98,112,120,140,94, 150,144,156,140,150,162

ii. 116,130,136,134,162,162,118,142,104,140,142,112,116,134,108,114,154,128, 116,140,122,122,172,128

$$\bar{x}_1 = \frac{\Sigma x}{n} = \frac{3338}{25} = 133.52$$

$$s_1 = \sqrt{\frac{\Sigma x^2 - (\Sigma x)^2/n}{n-1}} = \sqrt{\frac{460,748 - 445,689.76}{24}} = \sqrt{627.43} = 25.05$$

$$\bar{x}_2 = \frac{\Sigma x}{n} = \frac{3152}{24} = 131.33$$

$$s_2 = \sqrt{\frac{\Sigma x^2 - (\Sigma x)^2/n}{n-1}} = \sqrt{\frac{421,472 - 413,962.67}{23}} = \sqrt{326.49} = 18.07$$

b. The first set has the larger standard deviation: $25.05 - 18.07 = 6.98$.

c. The first set of observations is more dispersed than the second.

4.16 The median remains the same, 3.5, and the mean and standard deviation both become smaller.

4.18 a. $CV = \dfrac{21.21}{130.1} \times 100 = 16.3\%$

b. $CV = \dfrac{38.82}{216.96} \times 100 = 17.9\%$

c. The CV for blood pressure is somewhat less than the CV for cholesterol. The CV is a unit-free measure.

4.20 a. Negatively skewed

b. Positively skewed

c. Symmetric

4.23 $\sigma = \sqrt{\sigma^2} = \sqrt{144} = 12$

4.25 a. If the mean = median = mode, then the frequency distribution is symmetrical.

b. If mean = 15, median = 10, and mode = 5, then the frequency distribution would be positively skewed.
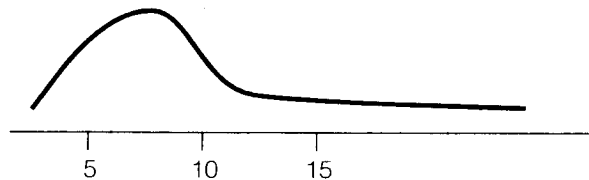
4.26 The sample mean $\bar{x} = \dfrac{\Sigma x}{n}$ is based on the sample size n and the population mean $\mu = \dfrac{\Sigma x}{N}$ is based on the entire population N.

Chapter 5

5.1 {TT, TH, HT, HH}

$P(0H) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$

$P(1H) = (\frac{1}{2})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2}) = \frac{1}{2}$

$P(2H) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$

5.3 {TTT, TTH, THT, THH, HTT, HTH, HHT, HHH}
a. $P(2H) = \frac{3}{8}$
c. $P(\text{at most } 2H) = \frac{7}{8}$

5.4 {GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB}
a. $P(2B + 1G) = \frac{3}{8}$
c. $P(0G) = \frac{1}{8}$
e. $P(2B \text{ followed by } 1G) = \frac{1}{8}$. Note that (a) does not consider order.

5.7 a. $P(\text{sum } 8) = \dfrac{5}{36}$

d. $P(\text{sum } 7 \text{ and both dice} < 4) = 0$

5.8 a. $P(O \text{ or } R) = \dfrac{10 + 15}{10 + 30 + 20 + 15} = \dfrac{25}{75} = \dfrac{1}{3}$

c. $P(\text{not } B) = \dfrac{55}{75} = \dfrac{11}{15}$

e. $P(R, W, \text{ or } B) = \dfrac{60}{75} = \dfrac{4}{5}$

5.9 $P(\text{white mouse in 10 hours}) = \dfrac{7}{10}$; $P(\text{black mouse in 10 hours}) = \dfrac{9}{10}$

a. $P(\text{both alive}) = \left(\dfrac{7}{10}\right)\left(\dfrac{9}{10}\right) = \dfrac{63}{100}$

b. $P(\text{black alive and white dead}) = \left(\dfrac{9}{10}\right)\left(\dfrac{3}{10}\right) = \dfrac{27}{100}$

d. $P(\text{at least one alive}) = \dfrac{7}{10} + \dfrac{9}{10} - \dfrac{63}{100} = \dfrac{97}{100}$

5.10 a. $P(\text{vegetarian}) = \dfrac{18 + 22}{18 + 22 + 20 + 23} = \dfrac{40}{83}$

c. $P(\text{male vegetarian}) = \dfrac{18}{83}$

5.11 a. $P(\text{completed high school}) = \dfrac{19}{100}$

c. $P(\text{physically inactive}) = \dfrac{49}{100}$

e. $P(\text{serum cholesterol} > 250; \text{systolic blood pressure} > 130) = \dfrac{9}{100}$

5.12 $5! = 120$

5.13 $P(10,4) = \dfrac{10!}{(10-4)!} = \dfrac{3,628,800}{720} = 5040$

5.15 $C(9,5) = \dfrac{9!}{5!(9-5)!} = \dfrac{362,880}{(120)(24)} = 126$

5.16   b. $C(6,4) = \dfrac{6!}{4!(6-4)!} = \dfrac{720}{24(2)} = 15$

   $P(n,r) > C(n,r)$ because order is considered.

5.17 $C(10,4) = \dfrac{10!}{4!(10-4)!} = \dfrac{3,628,800}{(24)(720)} = 210 = C(10,6)$

5.18   a. $P(3 \text{ out of } 5) = \dfrac{5!}{3!(5-3)!}\,(.5)^3(1-.5)^2 = \dfrac{120}{6(2)}\,(.5)^3(.5)^2$

   $= .3125$

   c. $P(\text{at most } 1) = P(0) + P(1) = .03125 + \dfrac{5!}{1!(5-1)!}\,(.5)^1(.5)^4$

   $= .03125 + .15625 = .1875$

   $n = 20; p = .25$

5.19   a. $P(3) = .1339$

   c. $P(< 3) = 1 - P(\geq 3) = 1 - .9087 = .0913$

5.20 $n = 10; p = .1$

   a. $P(10) = 0$

   c. $P(\geq 3) = 1 - (.3487 + .3874 + .1937) = .0702$

5.21 $n = 12; p = .25$

   a. $P(4) = .1936$

   c. $P(\geq 4) = 1 - .6488 = .3512$

5.22   10 males, 15 females; $P(\text{M smoke}) = \tfrac{1}{2}$, $P(\text{F smoke}) = \tfrac{1}{3}$

   a. $[P(4 \text{ of } 10 \text{ M smoke}) = .2051 \text{ and } P(6 \text{ of } 15 \text{ F smoke}) = .1786]$

   $P(4M \text{ and } 6F) = (.2051)(.1786) = .0366$

   c. $[P(0 \text{ of } 10 \text{ M smoke}) = .0010 \text{ and } P(0 \text{ of } 15 \text{ F smoke}) = .0023]$

   $P(0M \text{ and } 0F) = (.0010)(.0023) = .0000$

5.25   a. $P(A) = \dfrac{432}{4075} = .1060$

   b. $P(B) = \dfrac{768}{4075} = .1885$

   c. $P(B \mid A) = \dfrac{P(A \text{ and } B)}{P(A)} = \dfrac{42/4075}{.1060} = \dfrac{.0103}{.1060} = .0972$

   d. Because $P(B) \neq P(B \mid A)$ (that is, $.1885 \neq .0972$), events A and B are *not* independent.

5.27

Class Interval	Nonsmokers f_i	Smokers f_i	Total
90–109	10	5	15
110–129	24	15	39
130–149	18	10	28
150–169	9	3	12
170–189	2	2	4 ⎫
190–209	0	2	2 ⎬ 6
Total	63	37	100

a. $P(A) = \dfrac{63}{100} = .63$

b. $P(B) = \dfrac{37}{100} = .37$

c. $P(C) = \dfrac{6}{100} = .06$

d. $P(C\,|\,A) = \dfrac{P(C \text{ and } A)}{P(A)} = \dfrac{2/100}{.63} = .0317$

e. $P(C\,|\,B) = \dfrac{P(C \text{ and } B)}{P(B)} = \dfrac{4/100}{.37} = .1081$

The conditional probability of selecting someone with a blood pressure ≥ 170 from smokers is three times that of selecting someone from nonsmokers. Because $P(C\,|\,B) \neq P(C)$ (that is, $.1081 \neq .06$), smoking status and blood pressure are *not* independent.

Chapter 6

6.1 a. .4911   c. $2(.4678) = .9356$ e. .4990

6.2 a. $.5 - .4582 = .0418$ c. $5 - .4946 = .0054$ e. 0

6.3 a. 1.645   c. ± 1.96 e. ± 1.645

6.4 a. 1.645   c. 0

6.5 a. For 40%, Z_1 will be ± 1.282;
for 45%, Z_2 will be ± 1.645.
b. $x = \mu \pm Z\sigma = 130 \pm 1.282(17)$; $x = (108.2, 151.8)$

6.6 a. $Z_1 = (x - \mu)/\sigma = (45 - 60)/10 = -1.5$
$Z_2 = (75 - 60)/10 = 1.5$; area $= 2(.4332) = .8664 = 86.6\%$
c. $<50\ Z = (50 - 60)/10 = -1$; area $= .5 - .3413 = .1587 = 15.9\%$
e. $\geq 75\ Z = (75 - 60)/10 = 1.5$; area $= .5 - .4332 = .0668 = 6.68\%$

6.8 Mean $= 75$, $\sigma = 8$; 90th percentile; $Z = 1.28$
$1.28 = (x - 75)/8$; $x = (1.28)8 + 75 = 85.24 = 86\%$

6.9 Mean $= 50$, $\sigma = 12$

$P(x < 35) = (35 - 50)/12 = -1.25$; area $= .5 - .3944 = .1056$

6.10 a. The standard normal distribution has a mean $= 0$ and SD $= 1.0$. Other distributions have a variety of means and standard deviations.

b. Because the area is easily obtained for the standard normal distribution

6.12 a. $x = 55$; SD $= 6$

$$Z = \frac{65 - 55}{6} = \frac{10}{6} = 1.67$$

$P(Z) > 1.67 = .5 - .4525 = .0475$
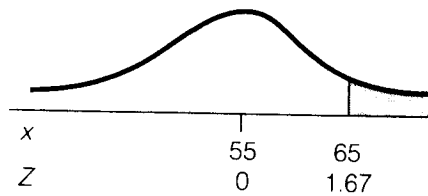
A little less than 5% will live another 65 years.

b. That life expectancy is normally distributed

6.15 Eliminating the 5% of students with the highest IQs and the 5% with the lowest IQs is equivalent to retaining students within $\pm 1.645\sigma$ of the mean; that is,
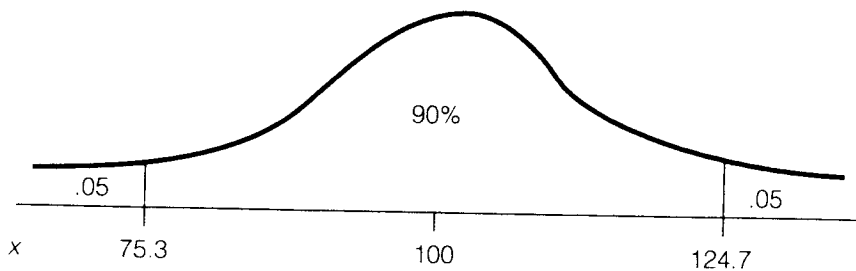
$$Z = \frac{x - 100}{15} \qquad Z(15) - 100 = x = 124.7$$

$$Z = \frac{x - 100}{15} \qquad Z(15) - 100 = x = 75.3$$

The lowest IQ for remaining students would be 75 and the highest IQ for remaining students would be 125.

6.17 $x = 4.7G$ SD $= .8G$

Prob (of pilot with $< 3.5G$)

$$\text{Prob}\left(\text{of } Z = \frac{3.5 - 4.7}{.8} = \frac{-1.2}{.8} = -1.5\right).$$
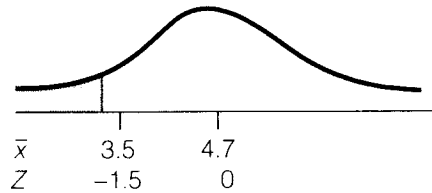
$$\text{Area} = .5 - .4332$$
$$= .0668$$

Chapter 7

7.1 $n = 36, \mu = 130, \sigma = 17$
Follows an approximately normal distribution with a mean equal to the population mean and a standard deviation of $\sigma/\sqrt{n}$

7.2 $n = 25, \mu = 60, \sigma = 10$
a. $P(57 < \bar{x} < 63)$; $Z_1 = (57 - 60)/(10/5) = -1.5, A = 2(.4332)$;
$Z_2 = (63 - 60)/2 = 1.5, A = .8664 = 86.6\%$
c. $P(x > 61)$; $Z = (61 - 60)/2 = 0.5$; $A = .5 - .1915 = .3085 = 30.9\%$

7.5 $\mu = 50, \sigma = 12$
a. $SE(\bar{x}) = \dfrac{12}{\sqrt{16}} = 3$
c. $SE(\bar{x})$ decreases when n increases.

7.6 $\mu = 71, \sigma = 5, n = 15$
a. $P(\bar{x} \geq 77)$; $Z = (77 - 71)/(5/\sqrt{15}) = 4.65$; $A = .999$
b. $P(65 < \bar{x} < 75)$
$$Z = \frac{65 - 71}{5/\sqrt{15}} = -4.65; A = .999$$
$$Z = \frac{75 - 71}{5/\sqrt{15}} = 3.099$$

7.7 $\mu = 52.5; \sigma = 4.5; P(\bar{x} > 56)$
a. $n = 10$; $Z = (56 - 52.5)/(4.5/\sqrt{10}) = 2.460$; $A = .0069$

7.8 $\mu = 3360, \sigma = 490$
a. $Z_1 = (2300 - 3360)/490 = -2.1633$,
$Z_2 = (4300 - 3360)/490 = 1.9184$; $A = .9572$
c. $Z = (5000 - 3360)/490 = 3.3469$; $A = <.001$

7.9 b. $Z_1 = (3100 - 3360)/(490/\sqrt{49}) = -3.7143$; $A \approx 1.0$;
$Z_2 = (3600 - 3360)/(490/\sqrt{49}) = 3.4286$
c. $Z = (2500 - 3360)/(490/\sqrt{49}) = -12.2857$; $A = 0$

7.12 a. The distribution of observations is more variable than the distribution of sample means. The distribution of sample means has the same mean as the parent distribution, but it has a smaller variance.
b. The standard deviation is a measure of variation of the individual's x's. The $SE(\bar{x})$ is a measure of variation of a sample of x's expressed as $\bar{x}$. Consequently, it is smaller.

c. In discussing the location of the individual x's, we would want to use the standard deviation. In trying to make inferences about the group (sample) mean, the x, we would want to use the SE(x).

7.14 $x = 220, \sigma = 50, n = 49$

$$\sigma_x = \frac{\sigma}{\sqrt{n}} = \frac{50}{\sqrt{49}} = \frac{50}{7} = 7.14$$

$$Z = \frac{200 - 220}{7.14} = \frac{-20}{7.14} = -2.80, \quad Z = \frac{240 - 220}{7.14} = \frac{20}{7.14} = 2.80$$

$$-2.80 \le P(Z) \le 2.8 = 2(.4974) = .9948$$

7.16 $\bar{x} = 2400, \sigma = 400, n = 64$

$$\sigma_x = \frac{400}{\sqrt{64}} = 50$$

a. $P(x) > 2500$

$$P(Z) = \frac{2500 - 2400}{50} = 2$$

$P(Z) > 2$ is $5 - .4772 = .0228$.

b. $2300 \le P(x) \le 2500$

$$-2 \le P(Z) \le 2$$

$$2(.4772) = .9544$$

c. $P(x) < 2350$

$$P(Z) < \frac{2350 - 2400}{50} = \frac{-50}{50} = -1.0$$

$$P(Z) < -1 = .5000 - .3413 = .1587$$

7.18 $\bar{x} = 128, \sigma = 12$

a. $122 < P(x) < 134$

$$\frac{122 - 128}{12} < P(z) < \frac{134 - 128}{12}$$

$$\frac{-6}{12} < P(z) < \frac{6}{12}$$

$$-.5 < P(z) < .5 = 2(.1915) = .3830$$

b. $122 < P(\bar{x}) < 134$

$$\frac{122 - 128}{12/\sqrt{16}} < P(Z) < \frac{134 - 128}{12/\sqrt{16}}$$

$$\frac{-6}{3} < P(Z) < \frac{6}{3}$$

$$-2 < P(Z) < 2 = 2(.4772) = .9544$$

c. The reason for the threefold difference in the probabilities of the events is that in (a) we are dealing with x—the individual blood pressure of a girl, and in (b) we are dealing with a much less variable entity: the x blood pressure based on a group of $n = 16$ girls.

7.20 $\bar{x} = 73, s^2 = 121$

a. $P(80 < x < 100); Z_1 = (80 - 73)/11 = .64; Z_2 = (100 - 73)/11 = 2.45;$
area $= .4929 - .2389 = .254$

c. $P(x > 90); Z = (90 - 73)/11 = 1.55;$ area $= .5 - .4394 = .0606$

7.24 $n = 100 \qquad x = 15 \qquad s = 40 \qquad SE(\bar{x}) = 4$

$$P(x < 160) \quad \text{or} \quad P\left(Z < \frac{160 - 150}{SE(x)} = \frac{10}{4} = 2.50\right)$$

$P(Z < 2.50) < .9938$

7.25 The $SE(\bar{x}) = \dfrac{s}{\sqrt{n}} = \dfrac{40}{\sqrt{100}} = \dfrac{40}{10} = 4$

7.27 One could use the central limit theorem to justify performing a test of hypothesis.

Chapter 8

8.1 $.64 - .51 \pm (1.96)(.17)\sqrt{(1/30) + (1/27)}$

$.04 < \mu_1 - \mu_2 < .22$

8.3 b. The 95% confidence intervals are as follows: $n = 25, 95\%$ CI $= 15.22 - 16.78;$ $n = 36, 95\%$ CI $= 15.35 - 16.65; n = 49, 95\%$ CI $= 15.44 - 16.56; n = 64, 95\%$ CI $= 15.51 - 16.49.$

c. Shrink. As the sample size increases (assuming a random sampling procedure was used) the confidence intervals should get closer and closer to the actual population mean.

8.5 $\mu = 200, \bar{x} = 225, \sigma = 16.67, n = 49, z = 1.96$

a. 95% CI of $\mu = 225 \pm (1.96)(16.67/\sqrt{49})$

$220.33 < \mu < 229.67$

$n = [(1.96)(16.67)]^2/10^2 = 10.67 \approx 11$

8.6 a. $\bar{x} = .33$

$.25 = \bar{x} - 3.00s$

$= .33 - 3.00s$

$s = .03$
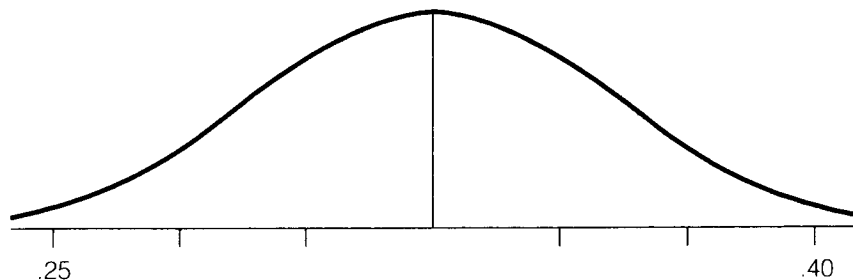
b. The 99% confidence interval formula for μ is 99% CI for $\mu = x \pm ts/\sqrt{n}$

$.33 \pm t(.03)/\sqrt{n}$

Exact values for CI can be determined for a known value of n.

8.9

	Mean	s	n	
Male	74.9	12.0	38	$s_p^2 = 131.51$
Female	71.8	11.0	45	

$$3.10 \pm 2.64(11.47)\sqrt{\frac{1}{38} + \frac{1}{45}}$$

$$-3.57 < \mu_1 - \mu_2 < 9.77$$

8.10 $\quad x_1 = 163.33 \qquad s_1 = 25.07$

$\bar{x}_2 = 179.90 \qquad s_2 = 33.87$

$$s_p = \sqrt{\frac{(25.07)^2(53) + (33.87)^2(50)}{54 + 51 - 2}} = 29.67$$

$$16.57 \pm 2.63(29.67)\sqrt{\frac{1}{54} + \frac{1}{51}}$$

$$1.33 < \mu_1 - \mu_2 < 31.81$$

8.12 a. $n = \left(\dfrac{Z\sigma}{d}\right)^2 = \left(\dfrac{2.57(1.6)}{0.5}\right)^2 = (8.22)^2 = 67.6 \quad \text{or} \quad 68$

b. $n = \left(\dfrac{Z\sigma}{d}\right)^2 = \left(\dfrac{1.96(1.6)}{0.5}\right)^2 = (6.27)^2 = 39.34$

8.14 $\quad$ 95% CI for $\sigma = 14.4 \pm 2.262\left(\dfrac{6.77}{\sqrt{10}}\right)$

$$= 14.4 \pm 2.262(2.141)$$
$$= 14.4 \pm 4.8$$
$$= 9.6 \text{ to } 19.2$$

8.16 a. 95% CI for $\mu_1 - \mu_2 = 262 - 236 \pm 2.01(49.5)\sqrt{\frac{1}{25} + \frac{1}{25}}$

$$= 26 \pm 2.01(49.5)(.2828)$$
$$= 26 \pm 28.1$$
$$= -2.1 \text{ to } 54.1$$

8.19 $\quad$ Male $x = 236 \qquad$ Female $x = 262$

$\qquad s_1 = 60 \qquad\qquad\quad s_2 = 64$

$\qquad n_1 = 25 \qquad\qquad\quad n_2 = 25$

$$s_p = \sqrt{\frac{s_1^2(n_1 - 1) + s_2^2(n_2 - 1)}{n_1 + n_2 - z}}$$

$$= \sqrt{\frac{60^2(24) + 64^2(24)}{25 + 25 - 2}}$$

$$= \sqrt{3848} = 62$$

The 95% CI for $\mu_1 - \mu_2 = \bar{x}_1 - \bar{x}_2 \pm 1.282\left(s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}\right)$

$$= 236 - 262 \pm 1.96\left(62\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}\right)$$

$$= -26 \pm 1.96\,(62)(.178)$$

$$= -26 \pm 1.96(11.048)$$
$$= -26 \pm 21.65$$
$$= 4.35 \text{ to } 47.65$$

So the 95% CI for $\mu_1 - \mu_2$ is 4.35 to 47.65.

8.21   The 99% CI for δ is

Treatment A	Treatment B	
$n_1 = 27$	$n_2 = 30$	
$\bar{x}_1 = 51\text{cc}$	$\bar{x}_2 = 64\text{cc}$	$s_p = -17$
$s_1^2 = 010$	$s_2^2 = .045$	

The 99% CI for $\mu_1 - \mu_2 =$

$$= \bar{x}_1 - \bar{x}_2 \pm Z_{(01)}(s_p)\sqrt{1/n_1 + 1/n_2}$$
$$= .64 - .51 \pm 2.58(.17)\sqrt{1/30 + 1/27}$$
$$= .13 \pm 2.58$$
$$= .13 \pm .4386(.2652)$$
$$= 13 \pm .4386$$
$$= 13 \pm .1163$$
$$= 0.0137 \text{ to } .2463$$

b. The 99% CI for $\mu_1 - \mu_2 = 262 - 236 \pm 2.797(49.5)(.2828)$
$$= 26 \pm 39.2$$
$$= -13.2 \text{ to } 65.2$$

Chapter 9

9.1   a. -1.645 or 1.645
      c. -3.012 and 3.012
      e. -2.03 and 2.03

9.2   a. Z test in (a) and (d)
      b. t test in (b), (c), and (e)

9.3   a. $H_0: \mu \le 30, H_1: \mu > 30$
      d. $H_0: \mu \ge 31.5, H_1: \mu < 31.5$
      e. $H_0: \mu = 16, H_1: \mu \ne 16$

9.4   a. -1.96 and 1.96
      c. -2.576 and 2.576
      e. -1.6759

9.6   a. Fail to reject H_0.
      c. Reject H_0.
      e. Fail to reject H_0.

9.7 $\mu = 85, n = 25, \bar{x} = 80.94$
      $Z(.05) = -1.645; t(.05, n - 1 = 24); H_0: \mu \ge 85, H_1: \mu < 85$
      a. $Z = \dfrac{80.94 - 85}{11.6/\sqrt{25}} = -1.750$ Reject H_0 and conclude that the boys were indeed
      underfed.

b. $t = \dfrac{80.94 - 85}{12.3/\sqrt{25}} = -1.650$ Fail to reject H_0 because $t = -1.650$ does not fall beyond the critical value $t = -1.711$.

9.9   Exercise 8.2: $Z = \dfrac{16 - 15}{2/\sqrt{25}} = 2.50$ $p = 2(.0062) = .0124$

Exercise 8.3: $Z = \dfrac{225 - 200}{16.67/\sqrt{49}} = 10.50$ $p = .0000$

Exercise 9.7a: $Z = -1.750$ $p = .0401$

Exercise 9.8: $t = 4.72$ $p < 2(.005) = .01$

Exercise 9.17: $t = \dfrac{73 - 70}{11.6/\sqrt{83}} = 2.36$ $p < .01$

9.11   a. $s_p = 11.55$; $\alpha = .05$; H_0; $\mu_v = \mu_{nv}$, H_1: $\mu_v \neq \mu_{nv}$

$t = \dfrac{73.5 - 72.9}{11.55\sqrt{(1/40) + (1/43)}} = .236$

Fail to reject H_0, and state the evidence was insufficient to indicate a difference in mean blood pressures between the two groups.

9.13 $\alpha = .05$; H_0: $\mu = 15$, H_1: $\mu \neq 15$ $z = (16 - 15)/(2/\sqrt{25}) = 2.50$
Two-tailed test: $z \pm 1.96$. Reject H_0, and conclude that the mean hemoglobin level is significantly different (higher) in this sample from that of the population mean.

9.15   a. paired t test   b. $\bar{x}_d = 275 - 260.6 = 14.4$
$t(.99, 9) = \pm 3.25$ H_0: mean difference between labs $= 0$
H_1: mean difference $\neq 0$

$s_d = \sqrt{\dfrac{2486 - (144)^2/10}{9}} = 6.77$

$t = \dfrac{14.4 - 0}{6.77/\sqrt{10}} = 6.73$

Reject H_0, and conclude that there is a significant difference between means of the two laboratories.
c. $t = (275 - 260.6)/20.62\sqrt{1/10 + 1/10} = 1.56$

9.17 $t = (73 - 70)/(11.6/\sqrt{83}) = 2.356$
H_0: $\mu \leq 70$, H_1: $\mu > 70$ $t(.01, 82, \text{one-tailed}) = 2.37$ (df 80 used)
Fail to reject H_0, and conclude that there is no significant difference in diastolic blood pressure.

9.19   a. $s_p = 29.67$
$t = (163.33 - 179.90)/29.67\sqrt{(1/54) + (1/51)} = -2.861$
$t(.05, 103, \text{one-tailed}) = -1.66$ H_0: $\mu_v - \mu_{nv} \geq 0$, H_1: $\mu_v - \mu_{nv} < 0$
Reject H_0 in favor of H_1, and conclude that the mean cholesterol level of vegetarians is significantly lower than that of nonvegetarians.

9.21   a. The variance is the same in both populations. The H_0 states that the two means are the same.
b. The basis for pooling the sample variances is that both populations are assumed to have the same variance.

9.23   a. 1. H_0: $\mu_1 - \mu_2 = 0$

2. $\alpha = .05$

3. $t = \dfrac{x_1 - x_2 - 0}{s_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \dfrac{133.14 - 125.24 - 0}{21.34\sqrt{\dfrac{1}{29} + \dfrac{1}{21}}}$

$= \dfrac{7.9}{21.34(.2865)} = 1.29$

4. Critical region is area beyond ± 2.01. (df = 48; df = 50 used).
5. Because the computed $t = 1.29 < 2.01$, the critical value, we fail to reject the H_0, and conclude that the blood pressure is not significantly different between smokers and nonsmokers.

b. 95% CI for $\mu_1 - \mu_2 = x_1 - x_2 \pm ts_p\sqrt{\dfrac{1}{29} + \dfrac{1}{21}}$

$= 7.9 \pm 2.045(21.34)(.2865)$
$= 7.9 \pm 12.5 = -4.6$ to 20.4

Yes, the decision reached would be the same: that there is no significant difference because zero is included in the interval. The decision has to be the same because both approaches use the same formula.

9.25 a. 1. $H_0: \mu_1 - \mu_2 = 0$
2. $\alpha = .05$

3. $t = \dfrac{262 - 236 - 0}{49.5\sqrt{\dfrac{1}{25} + \dfrac{1}{25}}} = \dfrac{26}{14.0} = 1.86$

4. The critical region is the area beyond ± 2.01 (df = 48; df = 50 used).
5. Because $t = 1.86$ falls in the fail-to-reject region, we fail to reject the H_0, and conclude that there is no significant difference at the $\alpha = .05$ level.

b. Because the 95% CI for $\mu_1 - \mu_2 = -2.9$ to 54.9 includes zero, there is no significant difference; that is, one reaches the same decision as in (a).

9.27 a. $H_0: \mu_1 = \mu_2 \qquad H_1: \mu_1 \neq \mu_2$.
b. df = 140, $\alpha = .05$, the critical value equals ± 1.98.
c. Independent
d. $t = 2.59$
e. H_0 should be rejected. There is evidence that the maximum daily alcohol consumption of college males is greater than that of college females.
f. 0.61–4.59

Chapter 10

10.1 a. $H_0: \mu_1 = \mu_2 = \mu_3$ (mean number of children is same for all groups)

b.

Source of Variation	SS	df	MS	F ratio
Between	381.67	2	190.84	26.84
Within	191.90	27	7.11	$F_{.95}(2,27) = 3.35$
Total	573.57	29		

c. Reject H_0, and conclude that the need for family planning counsel differs by the number of children per family.

10.3

Source of Variation	SS	df	MS	F ratio
Between	8,290.62	2	4,145.38	23.92
Within	3,118.33	18	173.24	$F_{.95}(2,18) = 3.55$
Total	11,414.95	20		

Reject H_0: $\mu_1 = \mu_2 = \mu_3$, and conclude that the mean ages of the three communities are different.

10.5 a. In a one-way ANOVA, one is able to partition the variation into two sources and test one of them. In a two-way ANOVA, one is able to partition the variation into three sources and test two of them.

b. That the observations are independent. Furthermore, that the observations of each group are normally distributed and that the variances of the various groups are homogeneous.

c. H_0: $\mu_1 = \mu_2 = \cdots = \mu_k$ for a one-way ANOVA

$\left.\begin{array}{l} H_0: \mu_1. = \mu_2. = \cdots = \mu_k. \\ H_0: \mu._1 = \mu._2 = \cdots = \mu._n \end{array}\right\}$ for a two-way ANOVA

10.7 a. For $\alpha = .05$: $F_{1,16} = 4.49$; $F_{3,16} = 3.24$; $F_{3,36} = 2.88$
b. For $\alpha = .01$: $F_{1,16} = 8.53$; $F_{3,16} = 5.29$; $F_{3,36} = 4.41$

10.9 a. $\text{HSD} = q(\alpha, k, N - k)\sqrt{\dfrac{\text{MSW}}{n}} = 3.53\sqrt{7.10/10} = 3.53(.843) = 2.97$

	1.7	7.7	10.2
1.7	—	6.0	8.5
7.7		—	2.5
10.2			—

Because only 6.0 and 8.5 exceed 2.97, they are the only significant pairs at $\alpha = .05$.

b. $\text{HSD} = 3.61\sqrt{173.24/7} = 3.61(4.975) = 17.96$

	23.0	71.3	41.9
23.0	—	48.3	18.9
71.3		—	29.4
41.9			—

Because all differences exceed the critical difference of 17.96, all pairs are significantly different from each other at the $\alpha = .05$ level.

10.11 a. $x_A = 23$; $x_B = 65.4$; $x_C = 37.8$

$\text{SS}_t = 35,483 - \dfrac{(715)^2}{18} = 7081.6$

$SS_b = 33,661.8 - 28,401.4 = 5260.4$

Source	SS	df	MS	F
Between	5260.4	2	2603.2	21.4
Within	1821.2	15	121.4	
Total	7081.6	17		

b. In MSB and the MSE, terms are both smaller, but the F statistic is still about the same. The three missing values made little difference on the overall outcome.

10.13   a. $SS_t = 8299 - 7980 = 319$ $SS_r = 319 - 116 - 143 = 60$

$SS_{tr} = 6(1352.01) - 7980 = 116$

$SS_b = 8123 - 7980 = 143$

Source	SS	df	MS	F
Treatment	132	2	66.0	13.75
Blocks	139	5	27.8	5.79
Residual	48	10	4.8	
Total	319	17		

b. Because $9.7 > F_{2,10} = 4.1$ at the $\alpha = .05$ level, there is a significant difference in the recidivism of the three programs.

c. Tukey's HSD is $q(\alpha, 3, 15)\sqrt{10/6} = 3.67(1.29) = 4.74$.

	24.9	19.8	18.5
24.9	—	5.0	6.3
19.8		—	1.3
18.5			—

All differences are significant except B and C at the $\alpha = .05$ level.

d. Because $4.8 > F_{5,10} = 3.33$ at the $\alpha = .05$ level, it appears that weight also is influential in recidivism.

10.14   A $-F$ ratio is not possible. There is an error in the calculations.

10.15   a. ANOVA Table

Source	SS	df	(MS) or s^2	F
Between	131.6	4	32.9	10.6
Within	94	30	3.1	
Total	225.6	34		

b. The calculated F is 10.6.

The critical $F_{.05}$ at df (4, 30) is 2.69.

The critical $F_{.01}$ at df (4, 30) is 4.02.

Your F ratio is significant at .05 and .01.

c. The q value from Appendix C is 4.10 for a .05 level of significance and 5.05 for a .01 level of significance.
The critical HSD for a .05 level of significance is 2.69, and for a .01 level of significance is 3.36.
There is a significantly greater weight gain in the following pairs at a .05 level of significance: AB, AC, AE, BC, CD, CE.
There is a significantly greater weight gain in the following pairs at a .01 level of significance: AC, CD.

Chapter 11

11.1 a. p_1 (none) = 25/100 = .25
p_2 (primary) = 32/100 = .32
p_3 (intermediate) = 24/100 = .24
$p_4 + p_5$ (high school and technical school) = 19/100 = 19
c. p_1 (mostly sitting) = .49; p_2 (moderate) = .51; p_3 (much) = 0

11.2 $\mu = np = 7683(.37) = 2842.71$

$\sigma = \sqrt{npq} = \sqrt{7683(.37)(.63)} = 42.32$

11.3 $H_0: p_H = .31$
$H_1. p_H \neq .31$
$\alpha = .05$
Critical region: $Z > 1.96, Z < -1.96$
$Z = \dfrac{\hat{p}_H - p}{\sqrt{pq/n}} = \dfrac{.37 - .31}{\sqrt{(.31)(.69)/100}} = 1.30$

Fail to reject H_0, and conclude that the evidence is insufficient to indicate that the proportion of smokers in Honolulu is signficantly different from that in the United States in general.

11.5 $p_1 = 4/7 = .57; p_2 = 7/21 = .33$

$p' = \dfrac{4 + 7}{7 + 21} = .39$

$SE(\hat{p}_1 - \hat{p}_2) = \sqrt{\dfrac{(.39)(.61)}{7} + \dfrac{(.39)(.61)}{21}}$

$= .213$

$H_0: p_1 - p_2 = 0$
$H_1: p_1 - p_2 \neq 0$
$\alpha = .05$

$Z = \dfrac{.57 - .33}{.213} = 1.127$

$Z(.025) = \pm 1.96$

Fail to reject H_0. There is no difference in the proportion of smokers between the two groups.

11.6 90% CI for $p_1 - p_2 = \hat{p}_1 - \hat{p}_2 \pm 1.645 \sqrt{\dfrac{(.57)(.43)}{7} + \dfrac{(.33)(.67)}{21}}$

$$= .24 \pm .35$$
$$= -.11 < \hat{p}_1 - \hat{p}_2 < .59$$

11.8 $\hat{p} = 29/99 = .293$

99% CI for $p = .293 \pm 2.576 \sqrt{\dfrac{(.293)(.707)}{99}}$

$$= .293 \pm .118$$
$$= .175 < p < .411$$

11.10 $\hat{p}_1 = \dfrac{55}{219} = .251; \hat{p}_2 = \dfrac{117}{822} = .142$

$\alpha = .01$

$p' = \dfrac{55 + 117}{219 + 822} = .165; q' = .835$

$SE(p_1 - p_2) = \sqrt{\dfrac{(.165)(.835)}{219} + \dfrac{(.165)(.835)}{822}} = .028$

$H_0: p_1 - p_2 = 0$

$Z = \dfrac{.251 - .142}{.028} = 3.893$ Critical region is ± 2.576 (for $\alpha = .01$).

Reject H_0, and conclude that there was a significantly higher proportion of those who started smoking at an earlier age among "abusers" than among "nonusers."

11.12 a. 99% CI for 11.10

$$p_1 - p_2 = \hat{p}_1 - \hat{p}_2 \pm 2.576 \sqrt{\dfrac{(.251)(.749)}{219} + \dfrac{(.142)(.858)}{822}}$$

$$= .109 \pm .082$$
$$= .027 < p_1 - p_2 < .191$$

11.13 a. $\mu = n\pi$ $\sigma = \sqrt{n\pi(1 - \pi)}$

b. p is the estimate of the parameter π.

c. The mean is $\dfrac{x}{n}$ and $\sigma_p = \sqrt{\dfrac{\pi(1 - \pi)}{n}}$.

d. when $n\pi \geq 5$ and $n(1 - \pi) \geq 5$ are satisfied

11.15 a. $p_1 = \dfrac{60}{100} = 0.6$ of males and $p_2 = \dfrac{70}{100} = 0.7$ of females

b. 99% CI for $\pi_1 - \pi_2 = p_1 - p_2 \pm Z \sqrt{\dfrac{p_1(1 - p_1)}{n_1} + \dfrac{p_2(1 - p_2)}{n_2}}$

$$= 0.6 - 0.7 \pm 2.58 \sqrt{\dfrac{.6(.4)}{100} + \dfrac{.7(.3)}{100}}$$

$$= -.1 \pm 2.58(.0671) = -.1 \pm .17$$
$$= -.27 \text{ to } .07$$

c. Since the CI for $\pi_1 - \pi_2$ includes zero in its interval, the difference is not a significant difference.

11.17 $p_1 = \dfrac{43}{100} = 0.43$ $p_2 = \dfrac{22}{100} = 0.22$

$p' = \dfrac{43 + 22}{100 + 100} = \dfrac{65}{200} = 0.33$

$SE(p_1 - p_2) = \sqrt{\dfrac{.33(.65)}{100} + \dfrac{.33(.65)}{100}} = \sqrt{.00429} = 0.0655$

1. $H_0: \pi_1 - \pi_2 = 0$ $H_1: \pi_1 - \pi_2 \neq 0$
2. $\alpha = .01$
3. $Z = \dfrac{p_1 - p_2 - 0}{SE(p_1 - p_2)} = \dfrac{.21}{.0665} = 3.16$
4. Critical region is area beyond 2.58
5. Because $3.16 > 2.58$, we reject the H_0 of equality and declare that geographics appears to play a signficant role in allergies.

11.19 $P = \dfrac{13}{100} = .13$ and 95% CI for $\pi = .13 \pm 1.96 \sqrt{\dfrac{P(1 - P)}{n}}$

$$= .13 \pm 1.96 \sqrt{\dfrac{.13(.87)}{100}}$$
$$= .13 \pm 1.96(.0336)$$
$$= .13 \pm .066$$
$$= 0.06 \text{ to } 20$$

Include a 95% CI for π and indicate that workers have a significant toxic exposure.

11.20 a. Proportions for Oregon: Before 0.29, after 0.24
Proportions for Washington: Before 0.28, after 0.29
b. Oregon: $p' = 1275 + 1023$ divided by $4475 + 4168 = .266$ SE $= .01$
$H_0: p_1 - p_2 = 0$
$H_1: p_1 - p_2 > 0$
$\alpha = .05$
$Z = \dfrac{.29 - .24}{.01} = 5.00$
$Z(.05) = 1.64$
Reject H_0. The proportion of fatally injured drivers, in Oregon, is lower after the enactment of the 0.08% law.
Washington: $p' = \dfrac{1735 + 1582}{6184 + 5390} = .287$ SE $= .008$
$H_0: p_1 - p_2 = 0$
$H_1: p_1 - p_2 > 0$
$\alpha = .05$

$$Z = \frac{.28 - .29}{.008} = -1.25$$

$Z(.05) = 1.64$

Fail to reject H_0. The proportion of fatally injured drivers, in Washington, is not lower after the enactment of the 0.08% law.

Chapter 12

12.1   a.

	Smokers		Nonsmokers	
	Observed	Expected	Observed	Expected
None	4	6.73	16	13.26
Primary	15	10.78	17	21.22
Intermediate	12	8.08	12	15.92
Senior high	1	3.03	8	5.97
Technical school	0	3.37	10	6.63

H_0: There is no association between smoking and educational level.

$\alpha = .05$

$\chi^2 = 14.17$

$\chi^2_{05}(df = 4) = 9.49$

Reject H_0.

b.

	Smokers		Nonsmokers	
	Observed	Expected	Observed	Expected
None	4	6.73	16	13.26
Primary	15	10.78	17	21.22
Intermediate	12	8.08	12	15.92
Senior high and technical school	1	6.40	18	12.60

$\alpha = .05$

$\chi^2 = 13.91$

$\chi^2_{05}(df = 3) = 7.81$

Reject H_0, and conclude that smoking is dependent on one's educational level—namely, smoking is less frequent among the more highly educated.

12.3   a.

	Egg Consumption							
	0		<1		2–4		Daily	
	O	E	O	E	O	E	O	E
Low	5	5.36	13	13.66	8	9.91	4	1.07
Medium	4	6.79	20	17.30	14	12.55	0	1.36
High	11	7.86	18	20.04	15	14.54	0	1.57

H_0: There is no association between egg consumption and age at menarche.
$\alpha = .05$
$\chi^2 = 14.59$
$\chi^2_{.05}(df = 6) = 12.59$
Reject H_0, and conclude that age at menarche is dependent on one's level of egg consumption.

b.

| | Egg Consumption | | | | | |
| | 0 | | <1 | | 2–7 | |
	O	E	O	E	O	E
Low	5	5.36	13	13.66	12	10.98
Medium	4	6.79	20	17.30	14	13.91
High	11	7.86	18	20.04	15	16.11

H_0: There is no association between egg consumption and age at menarche.
$\alpha = .05$
$\chi^2 = 3.26$
$\chi^2_{.05}(df = 4) = 9.49$
Fail to reject H_0: The data do not refute the H_0 of *no* association between age at menarche and egg consumption.

12.4 a.

| | Smoking | | No Smoking | | |
	O	E	O	E	Total
Hypertension group	4	2.75	3	4.25	7
Control group	7	8.25	14	12.75	21
Total	11		17		28

H_0: $p_1 = p_2$ (there is no difference in the proportion of smokers in the two groups).
$\alpha = .05$
$\chi^2 = 1.25$

$\chi^2_{.05}(df = 1) = 3.84$
Fail to reject H_0: The data do not refute the H_0 of no association.

12.5

| Age Interval | Heartbeat | | $O - E$ | $\dfrac{(O - E)^2}{E}$ | |
	O	E			
25–34	18	43.12	−25.12	14.63	
35–44	33	38.56	−5.56	0.80	$E_1 = \dfrac{140{,}195(188)}{611{,}152} = 43.13$
45–54	54	37.17	16.83	7.62	
55–64	48	30.42	17.58	10.16	
≥65	35	38.72	−3.72	0.36	
	188			33.57	

H_0: The age distribution of the heartbeat group is the same as that of the MSA.
$\alpha = .05$
$\chi^2 = 33.58$ and $\chi^2_{.05}(df = 4) = 9.49$
Reject H_0: The heartbeat age distribution is significantly different from the MSA age distribution.

12.7 a. The basis is the probability multiplication rule.
b. They are computed by multiplying the two marginal totals of the frequency and then dividing it by the total frequency.

12.9 The odds radio is $\dfrac{a/b}{c/d} = \dfrac{25/10}{14/51} = \dfrac{2.5}{.2745} = 9.1$

The risk of developing heart disease among smokers is 9.1 times that of non-smokers.

12.11 $RR = \dfrac{a/(a + b)}{c/(c + d)} = \dfrac{20/3299}{1/6701} = 40.6$

According to these data, the RR of developing lung cancer in smokers is 40.6 times that of nonsmokers.

12.13

	M	F	Total
Overweight	15	36	51
Not overweight	85	64	149
Total	100	100	200

$\alpha = .05$
H_0: Both groups are homegeneous

$$\chi^2 = \frac{200(15(64) - 36(85))^2}{100 \cdot 100 \cdot 51 \cdot 149} = 11.61$$

Conclusion: The two groups are not homogeneous at the .05 level because $\chi^2 = 11.61 > 3.84$.

12.15 a.

	M	F	Total
Belt	60	70	130
No belt	40	30	70
Total	100	100	200

$\alpha = .01$
H_0: Sexes are homogenous.

$$\chi^2 = \frac{200(60(30) - 70(40))^2}{100 \cdot 100 \cdot 130 \cdot 70} = 2.20$$

b. Because $\chi^2 = 2.20 < 3.84$, there is no significant difference in seat belt use between the sexes.

c. $\chi^2 = \dfrac{(60 - 65)^2}{65} + \dfrac{(70 - 65)^2}{65} + \dfrac{(40 - 35)^2}{35} + \dfrac{(30 - 35)^2}{35} = 2.20$. They are the same.

12.17

Substance Abuse	High	Low	Total
Alcoholic Family	28	12	40
Nonalcoholic Family	13	15	28
Total	41	27	68

$\alpha = .05$

$$\chi^2 = \frac{68((28)(15) - (13)(12))^2}{41 \times 27 \times 40 \times 28} = 3.82$$

Because $\chi^2 = 3.82 < 3.84$, there is no significant difference in juvenile substance abuse between alcoholic and nonalcoholic families.

Family Violence	Police Called to Home 1 or More Times	No Police Calls	Total
Alcoholic Family	25	15	40
Nonalcoholic Family	6	22	28
Total	31	37	68

$\alpha = .05$

$$\chi^2 = \frac{68((25)(22) - (15)(6))^2}{31 \times 37 \times 40 \times 28} = 11.2$$

Because $\chi^2 = 11.2 > 3.84$, there is a significant difference in the incidence of family violence between alcoholic and nonalcoholic families.

Neglect	Left Alone for Long Periods	Not Left Alone for Long Periods	Total
Alcoholic Family	5	35	40
Nonalcoholic Family	8	20	28
Total	13	55	68

$\alpha = .05$

$$\chi^2 = \frac{68((5)(20) - (35)(8))^2}{13 \times 55 \times 40 \times 28} = 2.75$$

Because $\chi^2 = 11.2 > 3.84$, there is no significant difference in the incidence of juveniles left alone between alcoholic and nonalcoholic families.

12.18

	Mutagen-Containing Meats						
	0–1 Servings		2–3 Servings		4 or More Servings		
Race	O	E	O	E	O	E	Total
African American	68	77.2	36	29.6	11	8.2	115
Whites	73	63.8	18	24.4	4	6.8	95
Total	141		54		15		210

H_0: There is no relationship between race and the consumption of mutagen-containing meats.

$\alpha = .05$

$\chi^2 = 7.60$

$\chi^2_{.05}(df = 2) = 5.99$

Reject H_0: There appears to be a relationship between race and the consumption of mutagen-containing meats.

12.19 a. The odds ratio is 3.8.

Chapter 13

13.1 a. Range -1 to 1

b. The sign tells the direction of the slope.

c. It tells the strength of the linear relationship.

d. It tells how good the prediction is likely to be.

e. Yes, they would have the same sign. No, they would not have the same magnitude.

13.2 a. $r = \dfrac{3,371,580 - (15,214)(21,696)/100}{\sqrt{2,611,160 - (15,214)^2/100} \cdot \sqrt{4,856,320 - (21,696)^2/100}}$

$= .336$

13.3 a. Plot of systolic blood pressure in row 3 (R3) versus cadmium level in row 1 (R1):
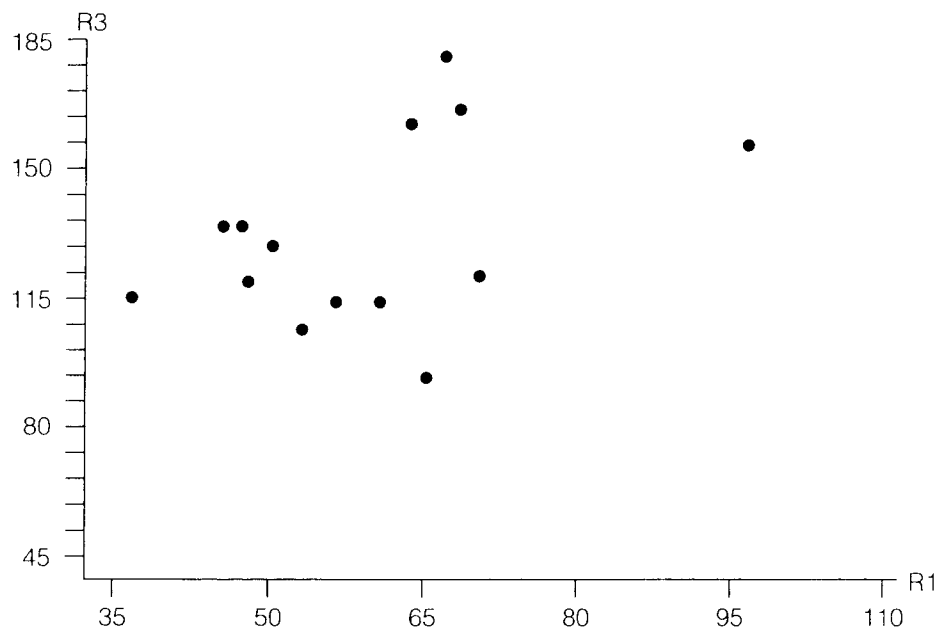
b. The plot does not support the notion that there is a strong linear relationship between the two variables.

c. Correlation of R1 and R3 $= 0.439$

d. Plot of zinc level in row 2 (R2) versus cadmium level in row 1 (R1):
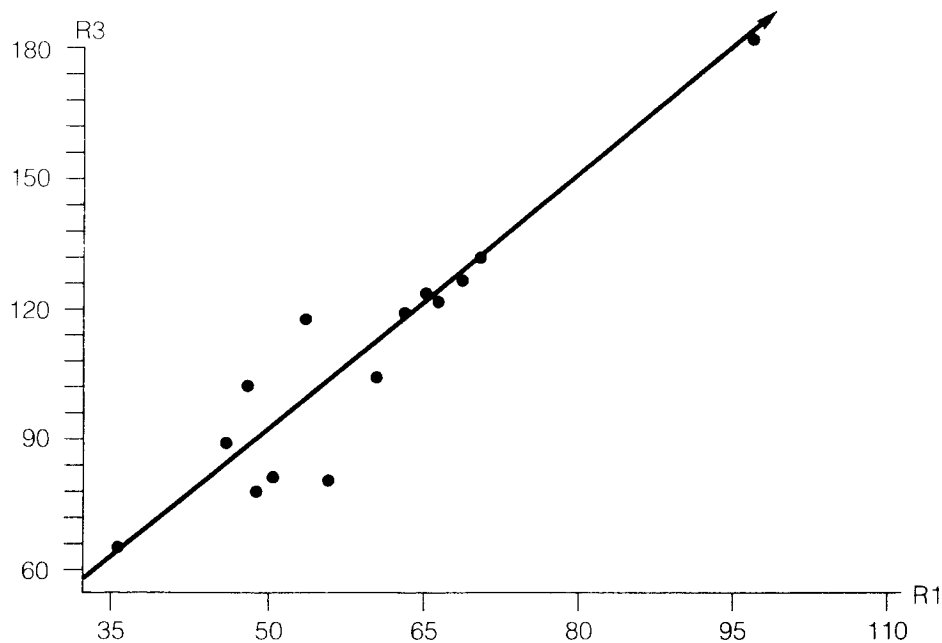
Correlation of R1 and R2 = .931

e. Yes, since it appears to follow a straight line.

f. $b = \dfrac{94{,}517 - (823)(1516)/14}{51{,}169 - [(823)^2]/14} = 1.936$

$a = 108.286 - (1.936)(58.786)$

$= -5.523$

$\hat{y} = 1.936x - 5.523$

g. $\hat{y} = 1.936(80) - 5.523 = 149.36$

h. No, because this is not possible to assess with this method.

13.5 a. $\alpha = .05; r = .447; n = 14; -.10 < \rho < .77$

b. Because $\rho = 0$ is included in the confidence interval, we fail to reject the H_0 of no relationship between cadmium and blood pressure.

13.6 $H_0: \beta = 0; \alpha = .05; b = 1.936$

$SE(b) = \sqrt{\dfrac{232.363}{46{,}364.357}} = .071; \quad t = \dfrac{1.936 - 0}{.071} = 27.3$

Reject H_0, and conclude that the population regression coefficient is significantly different from zero.

13.7 $b = \dfrac{3{,}371{,}580 - \dfrac{(15{,}214)(21{,}696)}{100}}{2{,}611{,}160 - \dfrac{(15{,}214)^2}{100}} = \dfrac{70{,}750.6}{296{,}502.1} = 0.2386$

$a = y - bx = \dfrac{21{,}696}{100} - .2386\left(\dfrac{15{,}214}{100}\right) = 217 - 36.3 = 180.7$

$$\hat{y} = 180.7 + .2386x$$

$$H_0: \beta_1 = 0 \quad \alpha = .01$$

$$t = \frac{b - 0}{SE(b)} = \frac{.2386}{.0656} = 3.64$$

$$s_{y \cdot x}^2 = \frac{\Sigma y^2 - a\Sigma y - b\Sigma xy}{n - 2}$$

$$= \frac{4,856,320 - 181(21,696) - .2386(3,371,580)}{98}$$

$$= 1274.3$$

$$SE(b) = \sqrt{\frac{1274.3}{296,502.1}} = \sqrt{.004298}$$

$$= .0656$$

Because $t = 3.64 >$ than the critical $t = 2.58$, we reject the H_0 in favor of $H_1: \beta_1 \neq 0$.

13.9   a. To test $H_0: \rho = 0$ we need to make the following assumptions:
       i. The pairs were obtained randomly.
       ii. x and y must be normally distributed.
   b. To test $H_0: \beta = 0$, we need to make the following assumptions:
       i.   The means of each distribution of y's for a given x fall on a straight line.
       ii.   The variances are homogeneous for each distribution of y's for each value of x.
       iii. The distribution of y's is normal for a given x.

13.11   The limitations are
     i. it measures only straight-line relationships
     ii. it does not prove a cause-and-effect relationship

13.13   a. $r = \dfrac{14,269.8 - \dfrac{(84)(2030)}{12}}{\sqrt{\left(589.2 - \dfrac{84^2}{12}\right)\left(346,940 - \dfrac{2030^2}{12}\right)}} = \dfrac{59.8}{\sqrt{(1.2)(3531.7)}} = \dfrac{59.8}{65.10} = 0.919$

     b.
95% CI for $\rho = .70$ to $.96$

13.15   b. $r = \dfrac{169,140 - \dfrac{(6640)226)}{11}}{\sqrt{\left(5,440,400 - \dfrac{6640^2}{11}\right)\left(6018 - \dfrac{226^2}{11}\right)}}$

$$= \frac{32,718}{\sqrt{(1,432,255)(1375)}} = \frac{32,718}{44,377} = 0.7373$$

The correlation coefficient of 0.73 is quite high, indicating a strong association of current death rates with cigarette consumption 20 years earlier.
$r^2 = .54$ provides an estimate of the total variation in y that is explained by the variation in x.

c. At $\alpha = .01, t = \dfrac{0.7373}{\sqrt{(1 - .54)/9}} = \dfrac{0.7373}{0.2261} = 3.26$

Because $t = 3.26 > 3.25$, the critical t value at the 1% level, there is a significant correlation between current death rates and previous cigarette consumption.

13.19 If you had 105 correlations, you would expect that by chance alone, 5% or approximately 5 of the correlations would be significant at the .05 level of significance.

13.20 $-.71$. The closer you are to either -1.00 or 1.00, the stronger the correlation. Conversely, the closer you are to 0.00, the weaker the correlation. The weakest correlation is .08.

13.22 a. $n = 25, .25-.80 \qquad n = 50, .39-.75 \qquad n = 100, .45-.71$

Chapter 14

14.1

	Breast-Fed			Not Breast-Fed	
No.	Age	Rank	No.	Age	Rank
1	14	10	1	9	4
2	15	11	2	10	5.5
3	12	7.5	3	8	3
4	13	9	4	6	1.5
5	19	12	5	10	5.5
			6	12	7.5
			7	6	1.5
			8	20	13

$W_1 = 49.5 \quad W_2 = 41.5$
$R_1 = 9.90 \quad R_2 = 5.2$

a. H_0: Breast-fed babies have more cavities than, or the same number of cavities as, non-breast-fed babies.

b. H_1: Breast-fed babies have fewer cavities than non-breast-fed babies.

c. $W_c = 35 \qquad \sigma_W^2 = 46.67; \ \sigma_W = 6.83 \qquad Z = \dfrac{W_1 - W_c}{\sigma_W} = \dfrac{49.5 - 35}{6.83} = 2.12$

One-tailed test: $Z(.05) = 1.64$

Reject H_0 in favor of H_1 and conclude that breast-fed babies have fewer cavities.

14.2 a. Vegetarians: $W_1 = 295; \ R_1 = 16.4; \ n_1 = 18$

Nonvegetarians: $W_2 = 408; \ R_2 = 21.5; \ n_2 = 19$

$W_e = \dfrac{18(18 + 19 + 1)}{2} = 342; \ \sigma_W = 32.9$

$Z = (295 - 342)/32.9 = -1.43$

Because $Z = 1.43$ is less than $Z(\alpha = .05) = 1.96$, we conclude there is no significant difference in diastolic blood pressure between the two groups.

14.3   a. H_0: The number of cavities for town A is the same as for town B.
  b. H_1: The number of cavities for the two towns is different.

  c.

Town A			Town B		
Person	# Cavities	Rank	Person	# Cavities	Rank
1	0	1	1	3	15
2	1	4	2	2	9
3	3	15	3	2	9
4	1	4	4	3	15
5	1	4	5	4	19.5
6	2	9	6	3	15
7	1	4	7	2	9
8	2	9	8	3	15
9	3	15	9	4	19.5
10	1	4	10	3	15
	$W_1 = 69$			$W_2 = 141$	
	$R_1 = 6.9$			$R_2 = 14.1$	

$W_e = 105$, $\sigma_W = 13.2$
  Since $z = -2.73$, which is less than -1.96, reject H_0.

$\Sigma r_d = 55$ $\Sigma r_{d(+)} = W_1 = 51.5$
$W_e = 27.5$ $\Sigma r_{d(-)} = W_2 = 3.5$

$$Z = \frac{51.5 - 27.5}{\sqrt{(20 + 1)27.5 / 6}}$$

Because $Z = 2.45$ is greater than 1.96, we reject H_0 in favor of H_1 and conclude that the two towns have different cavity levels; that is, the level is higher in the town with unfluoridated water.

14.5   Correlation of C11 and C22 = .736

$$t = \frac{736\sqrt{9}}{\sqrt{1 - (736)^2}}$$
$$= 3.262$$

Two-tailed test: $t(.975, 9) = 2.26$

a. H_0: There is no association between the cleanliness rankings of the two inspectors.
b. H_1: There is an association between the cleanliness rankings of the two inspectors.
  $\alpha = .05$
  Because $t = 3.262$ is greater than $t(.975, 9) = 2.26$, we reject H_0 and conclude that there is a high correlation between the cleanliness rankings of the two inspectors.

Row	Inspector (1) Column: C1 Count: 11	Inspector (2) C2 11	Inspector (1) Rank C11 11	Inspector (2) Rank C22 11
1	2.	1.	3.5	1.5
2	3.	3.	6.5	7.0
3	2.	3.	3.5	7.0
4	3.	2.	6.5	4.0
5	1.	2.	1.5	4.0
6	4.	5.	9.5	11.0
7	5.	4.	11.0	9.5
8	3.	2.	6.5	4.0
9	1.	1.	1.5	1.5
10	3.	4.	6.5	9.5
11	4.	3.	9.5	7.0

14.7 a. For the Wilcoxon rank-sum test, we need to be able to rank the combined distribution of two separate samples. We must be able to assign ranks to each of the observations and list and sum separately the ranks for the two samples.

b. For the Wilcoxon signed-rank test, we need a situation where we can obtain differences on each observation, as in a before-and-after situation. We then rank these differences according to the size of their absolute value, and then restore the original sign to each rank. There should be an equal number of positive and negative ranks if the H_0 is true.

c. For the Spearman rank-order correlation coefficient, we need to have two observations on each item observed. We then obtain these ranks separately for the x's and the y's. Next we obtain the differences on the ranks and square and sum them. The smaller the sum, the larger the coefficient.

14.9 a. $r_s = 1 - \dfrac{6(13.5)}{11(121 - 1)} = 1 - \dfrac{81}{1320} = 0.94$

b. H_0: There is no association between exercise and one's blood pressure.

c. H_1: There is an association between exercise and one's blood pressure.

$t = \dfrac{.94\sqrt{9}}{\sqrt{1 - .94^2}} = \dfrac{2.82}{0.34} = 8.3$

Because $t = 8.3 > t_{(.99)} = 3.25$ with 9 df, we reject the H_0 in favor of H_1 and conclude that the correlation coefficient is significantly different from zero.

14.11 a. for

A	B	Total
4	2	6
1	4	5
5	6	11

$P_1 = \dfrac{5!\, 5!\, 6!\, 6!}{11!\, 4!\, 4!\, 2!\, 1!} = \dfrac{5 \cdot 5 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 2} = \dfrac{25}{154}$

$P_1 = 0.162$

and for

A	B	Total
5	1	6
0	5	5
5	6	11

$$P_2 = \frac{5! \, 5! \, 6! \, 6!}{11! \, 5! \, 5! \, 0! \, 1!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{7 \cdot 8 \cdot 9 \cdot 10 \cdot 11} = \frac{1}{77}$$

$$P_2 = 0.013$$

so $P_1 + P_2 = 0.162 + .013 = 0.175$ and $2(.175) = .350$

a and b. H_0: The responses to A and B are the same.

$\alpha = 0.5$

Because $P = 0.35 > .05$, we do not consider the response to be significant.

14.13 $W_1 = 125.5; W_e = 143; Z = -1.36$

At a .05 level of significance, we fail to reject the null hypothesis. There is no difference between low-income African-American women and low-income white women, with respect to their consumption of meats.

Chapter 15

15.1 1970: 205.1 million
1980: 227.7 million
1990: 250.4 million

15.3 a. 196,000
b. 421,000

15.5

	Alaska	Kansas
Birthrate	21.8/1000	15.0/1000
Death rate	3.9/1000	8.9/1000

15.6 (1) Diseases of the heart; (2) malignant neoplasms; (3) cerebrovascular diseases; (4) accidents; and (5) chronic obstructive pulmonary diseases

15.7 1950: whites, 61.1 per 1000; nonwhites, 221.6 per 1000 (excludes Alaska and Hawaii)

15.9

	Deaths		
	Total	Infant	Neonatal
Riverside	8438	170	101
San Bernardino	8931	281	163

15.10 a. Alaska, 21.8 per 1000 population; Arizona, 18.7 per 1000 population

b. The birthrate in Alaska is higher because of the higher proportion of the population in the child-bearing age group.

15.11 The state with the highest birthrate in 1993 can be found in Table 93 of the *Statistical Abstract of the U.S. for 1996.*

The state with the highest birthrate is Utah, with 20.00/1000; and the highest fertility rate is for Utah, at 85.9.

15.12 The age-adjusted sex-specific death rate for cirrhosis of the liver can be found in Table 132. It is 18/1000 for men and 9.3/1000 for women.

15.13 The cause-specific death rates for 1993 for the three states and the United States can be found in Table 133 of the same reference as in 15.11.

	Cancer	Heart Disease	Diabeties	Accidents
Michigan	203.5	302.6	22.3	30.3
Utah	110.9	158.3	18.2	34.8
Tennessee	217	314.3	21.7	48.1
United States	205.6	288.4	20.9	18.8/1000

15.15 The three states in 1993 with the highest HIV death rates were (see Table 133): New York with 37.4, New Jersey with 28.0, and California with 20.3.

Chapter 16

16.1

Age Interval	l_x	d_x	$_nL_x$	T_x	$\hat{e}_x$
70–75	67,638	11,282	310,551	895,649	13.24
75–80	56,356	14,923	245,220	585,098	10.38
80–85	41,433	15,036	168,072	339,878	8.20

16.4 a. $_1q_0 = \dfrac{_1d_0}{l_0} = \dfrac{1260}{100,000} = .0126$

c. $_5q_0 = \dfrac{_5d_0}{l_0} = \dfrac{1260 + 257}{100,000} = .01517$

16.5 b. $_{10}q_{35} = \dfrac{_{10}d_{35}}{l_{35}} = \dfrac{851 + 1327}{95,641} = .02277$

16.6 a. $_{45}P_{20} = \dfrac{l_{65}}{l_{20}} = \dfrac{76,540}{97,668} = .7837$

16.7 a. At birth, $\hat{e} = 73.62$.

c. At 35 years, $\hat{e} = 41.25$.

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Index

Table A Areas Under the Normal Curve

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4350	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

ea = 0.20	0.40	0.60	0.80	0.90	0.95	0.98	0.99	0.999

Probability Below *t* Value (One-Sided)

ea = 0.60	0.70	0.80	0.90	0.95	0.975	0.99	0.995	0.9995
0.3250	0.7270	1.376	3.078	6.3138	12.706	31.821	63.657	636.619
0.2885	0.6172	1.061	1.886	2.9200	4.3027	6.965	9.9248	31.598
0.2766	0.5840	0.978	1.638	2.3534	3.1825	4.541	5.8409	12.924
0.2707	0.5692	0.941	1.533	2.1318	2.7764	3.747	4.6041	8.610
0.2672	0.5598	0.920	1.476	2.0150	2.5706	3.365	4.0321	6.869
0.2648	0.5536	0.906	1.440	1.9432	2.4469	3.143	3.7074	5.959
0.2632	0.5493	0.896	1.415	1.8946	2.3646	2.998	3.4995	5.408
0.2619	0.5461	0.889	1.397	1.8595	2.3060	2.896	3.3554	5.041
0.2610	0.5436	0.883	1.383	1.8331	2.2622	2.821	3.2498	4.781
0.2602	0.5416	0.879	1.372	1.8125	2.2281	2.764	3.1693	4.587
0.2596	0.5400	0.876	1.363	1.7939	2.2010	2.718	3.1058	4.437
0.2590	0.5387	0.873	1.356	1.7823	2.1788	2.681	3.0545	4.318
0.2586	0.5375	0.870	1.350	1.7709	2.1604	2.650	3.0123	4.221
0.2582	0.5366	0.868	1.345	1.7613	2.1448	2.624	2.9768	4.140
0.2579	0.5358	0.866	1.341	1.7530	2.1315	2.602	2.9467	4.073
0.2576	0.5358	0.865	1.337	1.7459	2.1199	2.583	2.9208	4.015
0.2574	0.5344	0.863	1.333	1.7396	2.1098	2.567	2.8982	3.965
0.2571	0.5338	0.862	1.330	1.7341	2.1009	2.552	2.8784	3.922
0.2569	0.5333	0.861	1.328	1.7291	2.0930	2.539	2.8609	3.883
0.2567	0.5329	0.860	1.325	1.7247	2.0860	2.528	2.8453	3.850
0.2566	0.5325	0.859	1.323	1.7207	2.0796	2.518	2.8314	3.819
0.2564	0.5321	0.858	1.321	1.7171	2.0739	2.508	2.8188	3.792
0.2563	0.5318	0.858	1.319	1.7139	2.0687	2.500	2.8073	3.767
0.2562	0.5315	0.857	1.318	1.7109	2.0639	2.492	2.7969	3.745
0.2561	0.5312	0.856	1.316	1.7081	2.0595	2.485	2.7874	3.725
0.2560	0.5309	0.856	1.315	1.7056	2.0555	2.479	2.7787	3.707
0.2559	0.5307	0.855	1.314	1.7033	2.0518	2.473	2.7707	3.690
0.2558	0.5304	0.855	1.313	1.7011	2.0484	2.467	2.7633	3.674
0.2557	0.5302	0.854	1.311	1.6991	2.0452	2.462	2.7564	3.659
0.2556	0.5300	0.854	1.310	1.6973	2.0423	2.457	2.7500	3.616
0.2553	0.5292	0.8521	1.3062	1.6896	2.0301	2.438	2.7239	3.5919
0.2550	0.5286	0.8507	1.3031	1.6839	2.0211	2.423	2.7045	3.5511
0.2549	0.5281	0.8497	1.3007	1.6794	2.0141	2.412	2.6896	3.5207
0.2547	0.5278	0.8489	1.2987	1.6759	2.0086	2.403	2.6778	3.4965
0.2545	0.5272	0.8477	1.2959	1.6707	2.0003	2.390	2.6603	3.4606
0.2543	0.5268	0.8468	1.2938	1.6669	1.9945	2.381	2.6480	3.4355
0.2542	0.5265	0.8462	1.2922	1.6641	1.9901	2.374	2.6388	3.4169
0.2541	0.5263	0.8457	1.2910	1.6620	1.9867	2.368	2.6316	3.4022
0.2540	0.5261	0.8452	1.2901	1.6602	1.9840	2.364	2.6260	3.3909
0.2539	0.5258	0.8446	1.2887	1.6577	1.9799	2.358	2.6175	3.3736
0.2538	0.5256	0.8442	1.2876	1.6558	1.9771	2.353	2.6114	3.3615
0.2538	0.5255	0.8439	1.2869	1.6545	1.9749	2.350	2.6070	3.3527
0.2537	0.5253	0.8436	1.2863	1.6534	1.9733	2.347	2.6035	3.3456
0.2537	0.5252	0.8434	1.2858	1.6525	1.9719	2.345	2.6006	3.3400
0.2533	0.5244	0.8416	1.2816	1.6449	1.9600	2.326	2.5758	3.2905